**Bounded Rationality, Heterogeneous Beliefs and the Evolution of the Economy.**

Jens Peter Soerensen

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Declaration.

This thesis consists entirely of my own original work and has been composed by myself.

Signed :

**Contents.**

**Abstract.**

In this thesis I study the implications of the speed of learning in an overlapping generations model. The equilibrium can be altered, for example, by a change in the level of government purchases and this can be financed in various ways. The purpose is to examine the effects on a number of important economic variables including welfare during the learning transition to the new steady state. Chapter 1 contains the introduction. In chapter 2 there is a comparison between an economy in which all agents are fast learners with an economy in which all agents are slow learners. Under certain conditions it is shown that while welfare may initially be higher in an economy with fast learners, this will not continue to be the case during the whole learning transition. After some time, possibly quite early, welfare will be higher in an economy populated by slow learners. This analysis is extended to a model with externalities and coordination failures in chapter 3 and models with random productivity and preference shocks are studied in chapter 4.

In chapter 5, I consider the consequences of agents having heterogeneous expectations, i.e. slow and fast agents in the same economy. The analysis considers two issues: (i) under what conditions is convergence of learning guaranteed when there is heterogeneous learning and (ii) how does welfare compare for fast relative to slow learners, when there is a mixture of the two types. The stability analysis extends earlier results in the literature for the homogeneous case. The welfare comparison gives a much more intuitive result. The welfare of the fast agents is higher than the welfare of the slow agents during the entire learning transition. This does not depend on whether one is looking at expansionary or contractionary monetary policy, providing that the economy started in equilibrium. In an extension of this analysis, I study the case where there are more than 2 classes of agents.

In chapter 6, the convergence with heterogeneous expectations when there equilibrium 2-cycles are analysed and an equivalence result between stability in the homogeneous case and the heterogeneous case is established. Chapter 7 provide summary and conclusions.

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# Chapter 1. Introduction.

There has been an increased interest in the possibility of endogenous fluctuations in dynamic macroeconomic models[1]. The rational expectations approach has been used widely as the equilibrium concept in these models, and this procedure has been very successful, but we are left with two questions. First, how can the rational expectations solution be attained if the agents do not begin with rational expectations and second, which rational expectations solution will the economy follow if there are multiple rational expectational solutions. In order to answer these questions there has been a widespread attention into the convergence of adaptive learning rules to the rational expectations equilibrium, see Evans and Honkapohja (1995c) for a recent overview of the literature. Lucas (1986) suggests that a plausible learning process should converge to the rational expectations equilibrium where money has value, furthermore Evans (1989) together with Evans and Honkapohja (1995a and 1995b) suggests that in the case of multiple rational expectations equilibria, a simple adaptive learning rule will track down locally unique rational expectations solutions.

An important distinction in the literature is made between linear models and nonlinear models, where learning in linear models has been analysed by Bray (1982), Bray and Savin (1986), Marcet and Sargent (1989a and 1989b), Evans and Honkapohja (1994a) and others. Learning in nonlinear models has been considered by Bullard (1994), Duffy (1994) Evans and Honkapohja (1993a,1994b, 1995a,b,c and d), Grandmont (1985), Grandmont and Laroque (1986, 1991), Guesnerie and Woodford (1991, 1992), Sargent (1992) and Woodford (1990).

In this thesis we will focus on learning in nonlinear models. A large part of the literature has been concerned with finding a answer to the two questions mentioned above, and simple adaptive learning rules has been suggested as a selection criteria to pick out rational expectations equilibria. However there has recently been some

[1] See e.g. Grandmont (1985) and the recent survey by Guesnerie and Woodford (1992).

criticism of this approach that a plausible learning rule should pick the relevant locally unique solutions. Woodford (1990) shows that a learning process may converge to a sunspot solution instead of the rational steady state, Evans and Honkapohja (1994b) study the local stability of sunspots under learning as well. Grandmont and Laroque (1991) is very critical of some of the assumptions needed for many adaptive learning rules to converge to the rational expectations solution, while Duffy (1994) shows local convergence of an adaptive learning rule to an indeterminate steady state.

A common feature of a large number of these papers is that agents are assumed to use the same learning rule, such that the agents are homogeneous in their learning. The contribution of this thesis is investigate what happens in the economy, when the agents have different learning rules, in the sense that some agents have a *fast* learning rule and others have a *slow* learning rule. The definition of fast and slow agents will be defined more precisely below. Hence in this thesis we will introduce heterogeneity of the agents learning rules, and analyse the effects on the evolution of the economy during a learning transition towards a rational expectations equilibrium. This is done for different cases within the context of a standard overlapping generations model. We wish to look for a role for government policy in these models, and how the stability conditions under learning is change when there is heterogeneity in the model. Another aspect is the behaviour of welfare during the learning transition to rational expectations solution, is it possible that agents with a poorer forecast can do better in terms of welfare than agents with a better forecast. This would be interesting when the government use a mixture fiscal and monetary policy to move the economy between equilbria, in order to the see effect of learning, especially if agents with different forecast affect each other.

Heterogeneous beliefs has not been considered very often in the literature, although it should be an obvious extension of the model to allow the agents to have different beliefs. One reason might be that the models under consideration can be *generalised trivially* to the case in which the expectation formation of each agent is different and it does not change the results, as mentioned in footnotes by Grandmont (1985), and Grandmont and Laroque (1986). However Duffy (1994) gives the opposite reason for letting all agents having the same learning rule, since the "*analysis of the case where agents have heterogeneous beliefs is especially difficult*" as mentioned by Duffy.

In Bray and Savin (1986) there is an analysis of convergence of heterogeneous expectations to the rational expectations solution by introducing different priors in the context of the cobweb model, but the learning rules of different suppliers are identical. Marcet and Sargent (1989b) analyse some situations of heterogeneity. In Evans, Honkapohja and Sargent (1993), there is combination of "econometricians" and perfect foresight agents, and this used to study the existence of 2-cycles and to investigate whether the fraction of econometricians can affect the existence of a 2-cycle. A recent paper by Evans, Honkapohja and Marimon (1995) study convergence in a overlapping generations model with money with heterogeneous agents.

We will concentrate on nonlinear models of the form

(1.1) $$x_t = F(x^e_{t+1})$$

where F is some continuously differentiable function such that the actual value $x_t$ depend on the forecast $x^e_{t+1}$, see for example Grandmont and Laroque (1986) and Woodford (1990). In this case we assume that the agents have point expectations, and the model is purely deterministic, this is the simplest version which will be used in chapter 2 and 3. We can easily extend the model to more general versions such as $x_t = \{F(x_{t+1})\}^e$ or $x_t = \{F(x_{t+1})\}^e + v_t$, where $v_t$ is a random variable. These models can all be included within the framework studied in Evans and Honkapohja (1995a), where we have the following model

(1.2) $$x_t = H(G(x_{t+1}, v_{t+1})^e, v_t).$$

In this situation the actual value depend on the forecast of $G(x_{t+1}, v_{t+1})$ as well as the random variable $v_t$.

The nonlinear model given in (1.1) typically arise from a standard overlapping generations model, which is deterministic. However in many cases the overlapping generations model with some form of randomness included requires the more general form (1.2). The random variable can, for example, be a productivity shock or a preference shock, but many other specifications shocks are possible. In general the F-function from (1.1) can be increasing, decreasing or hump-shaped depending on the preferences of the underlying model. The different shapes of F can give other types of rational expectations equilibria apart from steady states, for example, equilibrium cycles and sunspot equilibria, together with more exotic equilibria, see Bullard (1994)

The learning rule we will consider in this thesis is a simple adaptive learning rule of the form

(1.3) $$x^e_{t+1} = x^e_t + a_t(x_{t-1} - x^e_t) \qquad \text{for all t.}$$

where $a_t$ is *the speed of learning*, this is sometimes called the gain parameter. This learning rule is referred to as econometric learning or adaptive learning. The idea is that agents use some perceived law of motion to calculate their forecasts of the variables of interest, and use some standard statistical procedure to estimate it. The forecast for the variables of interest are then calculated from the estimated law of motion, this is referred to as reasonable learning schemes by Bray (1982). The learning rule (1.3) has the merit of simplicity and there has been an extensive use of it. When the agents use (1.3) they are not "fully rational" since the agents use a model that are misspecified during the learning transition, in the sense they are outside the rational expectations solution. As Woodford (1990) remarks "*Indeed, no model that that seriously attempts to model behaviour under an assumption of total ignorance about the equilibrium behaviour of prices can avoid being unsatisfactory in this respect.*" There has been some attempts to model "rational learning", but this will not be considered in this thesis.

There are some papers that discuss the role of the speed of learning, $a_t$. Guesnerie and Woodford (1991), analyses the local stability of equilibrium k-cycles and how the stability conditions depend on the size of $a_t$. Vives (1993) analyse, what factors influence the speed of learning, and given that the learning process converge to a rational expectations solution, Vives find a number for the rate of convergence. There are other types of learning rules agents could use, some of these rules are similar to (1.3) in the sense that agents are boundedly rational, see e.g. Woodford (1990) and Sargent (1993) for a description of a gradient rule. Grandmont and Laroque (1986) use a learning rule with finite memory which is time-invariant.

The learning rule (1.3) is formulated for agents who believe they are in steady state and formulate a learning rule according to these beliefs. If the agents, for example, believe the economy where in a k-cycle, or a sunspot equilibrium, they would formulate a learning rule according to these beliefs. The question is whether the steady state is locally stable under a variation in the learning rule, if the agents believe the economy is in a more exotic equilibrium. In fact, the local stability results are quite sensitive to variations in the variables used in the learning rule. Evans (1989) suggests, that we should distinguish between weak stability and strong stability, where weak stability of the rational expectations solution is defined as local stability under the learning rule (1.3), while the rational expectations solution is strongly stable if it is locally stable under an overparameterised learning rule. Weak and strong stability of steady states and sunspots are considered in Woodford (1990), and Evans and Honkapohja (1994b), while weak and strong stability of cycles is considered in Evans and Honkapohja (1995a). In this thesis we will focus on the learning rule (1.3) and therefore mainly consider weak stability.

The definition of fast and slow agents is related to the speed of learning given in the learning rule (1.3). Let the fast agent's speed of learning be given by $a_t^f$ for all t, and let the slow agent's speed of learning be denoted by $a_t^s$ for all t, then the fast agents have a higher speed of learning at each time t.

(1.4) $\qquad a_t^f > a_t^s \qquad$ for all t.

Hence fast and slow are not meant as the fast agents learning more than the slow agent, the fast agents are just closer to the actual value of the variables of interest. Here both types of agents are boundedly rational and they do not "become" rational during the learning transition. We will look at different ways of incorporating fast and slow learners in the overlapping generations model. In the first part of the thesis we will compare two economies, one economy with fast learners and one economy with slow learners. There are no interaction between the two economies, but we can show in some cases that the slow learners are better off in terms of welfare compared to fast learners level of welfare during a part of the learning transition, but in other cases fast learners is better off. A natural extension is to have a mixture of fast and slow agents in the same model, this changes the F-function in (1.1) and makes the model more complicated. This is studied in the last part of the thesis.

The thesis is organised as follows. In chapter 2, we outline the basic overlapping generations model. There are no externalities, for example, increasing returns in the production function, and the model is purely deterministic. The maximisation problem for the agents give a reduced form like (1.1). An agent make a forecast of the price in the following period, and it turns out that the agent might as well forecast the labour supply according to (1.3). Hence the dynamic system can be described by (1.1) and (1.3). In this very simple model there is a unique steady state, which is globally stable under the learning rule (1.3), given certain assumptions on the functions. We analyse the evolvement of labour supply, prices and welfare during the movement to the steady state. We compare an economy with slow learners to an economy with fast learners, and show that the welfare of a slow agent born at time t is higher than the welfare of fast agent born at time t, when t is large and the initial forecast is below the steady state. This result occurs although the fast agent is is "closer" to the steady state. This learning effect depend partly on the fact that there is no interaction between slow and fast agents. If the initial forecast is above the steady state we have the opposite result when t is large. We introduce a government that use different types of policies to change the steady state. In this simple set-up there is no role for government intervention, the different policies are purely distortionary and are used to create "fluctuations", and we can use the welfare analysis above to compare welfare between fast and slow learners.

In order to give a role for government policy, the model from chapter 2 is extended to include increasing social returns, where we use a production function which exhibits increasing social returns. The microfoundations for this function is developed in Evans and Honkapohja (1995b). The increasing social returns give rise to multiple equilibria with possible coordination failures. The government use a subsidy to the price of production to push the economy from a low level of activity equilibrium to a high level of activity equilibrium. The welfare comparison in this situation become more clear-cut in some cases, since the fast learners can be better off during the whole learning transition to the high level equilibrium. However there are still cases in which the slow agents are better off when t is large, even when the steady state welfare is rising, thus the learning effect is still existing in this model.

In addition to the learning effect there are two other effects present, there is the effect of having increasing returns, which increase the steady state welfare when moving from a low-activity steady state to a high activity steady state. Furthermore, if the increase in the subsidy is too large, the welfare in the new high-activity steady state can decrease compared to the welfare in the previous high-activity steady state, hence there is an "optimal" steady state subsidy. We briefly discuss the problem of finding an "optimal" path for the subsidy in end of chapter 3.

Chapter 4 introduce randomness in the model from chapter 2 and 3. First, a very simple type of preference shock is incorporated in the model from chapter 2, similar to Woodford (1990), this is done in order to investigate whether the welfare comparison changes, since we might expect the welfare comparison to change with a shock in the model. In this simple case the steady state does not depend on the distribution of the shock, as it would in more general cases, for example, in the reduced form (1.2). In order to show local stability of the steady state under learning, we assume that the speed of learning $a_t$ is decreasing over time :

$$a_t \rightarrow 0 \quad \text{when } t \rightarrow \infty$$

If $a_t = a$ is constant for all t, a large shock early on might push the economy "away" from steady state. The stability conditions for local stability corresponds to the expectational stability conditions, see Evans and Honkapohja (1995a and 1995c).

We continue by incorporating increasing returns, together with a more "interesting" shock to the model. There are many types of preference, monetary or productivity shock we could incorporate, but here we use a productivity shock. In this case each of the steady states from chapter will correspond to a "noisy" steady state. The existence and stability of the noisy steady state depends on results from Evans and Honkapohja (1995a). It is possible to obtain fluctuations between a low-level steady state and a high-level steady state, without any government intervention or any other structural changes, if the speed of learning $a_t$ is constant, $a_t = a$ for all t. The reason is that for t large, the agents still produce a noisy forecast, while in the case of decreasing speeds, the forecast becomes constant when t is large. If we compare an economy with slow learners with an economy of fast learners, the welfare results does not hold when t is large and the two groups of agents use a constant speed of learning. This is due to the fact that both types of agents produce noisy forecast for all t. The reason for having a constant speed of learning is to be able to track structural changes, or changes in policy, much more rapidly than can be done with a decreasing speed of learning.

In the previous chapters there was no interaction between the two groups of agents, this assumption is changed in chapter 5. In this chapter, we have a combination of fast and slow learners in the overlapping generations model from chapter 2, or to be precise we have a fraction $\mu$ fast learners and a fraction $1 - \mu$ slow learners. This change the function F from (1.1), and instead we have the reduced form :

(1.5) $$x_t = H(x_{t+1}^{f,e}, x_{t+1}^{s,e}, \mu)$$

where the actual value $x_t$ at time t depends on the fast learners expectations $x_{t+1}^{f,e}$ and the slow learners expectations $x_{t+1}^{s,e}$. The agents forecast the price instead of the labour supply, since it is technically easier. In this situation we have two non-linear difference equations, and the local stability of the steady state under learning depend on the fraction of learners, $\mu$. The welfare comparison in this case is simpler since both groups of agents face the same actual price. Given a concavity assumption on H, we can show that the fast agents are better off than the slow learners during the learning transition to the steady state when both groups have the same initial forecast. This is due to the fast agents being closer to the actual price. Hence the fast agent is "less" misspecified than the slow agent. However, if the agents do not have the same initial forecast, then we can show with the help of simulations that the slow learners might be better off during a part of the learning transition to the steady state. In the last section the model is extended to included n learners, but this does not change the results.

In chapter 6 we study the local stability of cycles under learning in the model from chapter 5. The increased number of difference equations makes the analysis somewhat more complicated than models with one type of agents. Models with homogeneous agents have been studied by for example by Grandmont and Laroque (1986), Guesnerie and Woodford (1991), and Evans and Honkapohja (1995a). We will look at the situation where we have decreasing speeds of learning, but there is no shock in the model, it is purely deterministic. The stability analysis is done by looking at a corresponding differential equation instead of the difference equation. This is similar to Ljung (1977), however his difference equation is stochastic, thus we reformulate his difference equation to our problem. We find a condition that ensures local stability under learning. When we have heterogeneous agents, the conditions for local stability of a 2-cycle under learning will actually depend on the stability conditions for the homogeneous case. Chapter 7 contains the conclusions and some remarks on how we might possibly extend the results from the previous chapters.

## Chapter 2. Fast learners versus slow learners.

### 2.1. Introduction.

In this chapter we consider a standard overlapping generations model with money, similar to a model in Evans and Honkapohja (1995b). The agents have an adaptive learning rule, which they use to make their forecast of the relevant variable, for example, the price or the labour supply. There is only one interior steady state in this simple model, but the model may have other more exotic equilibria. We introduce a government that uses different combinations of fiscal and monetary policy to change the steady state.

The purpose is to analyse the welfare consequences during the movement to a steady state, where the agents are using an adaptive learning rule to make a forecast of the labour supply. We compare two separate economies, where the difference between the economies are in the agents speed of learning. We define two types of learners, fast learners and slow learners, where the fast learners have a high speed learning rule and the slow learners have a low speed learning rule. Since both groups use an adaptive learning rule, they are not "fully" rational, but the fast learner make a better forecast compared to the slow learner as a result of higher speed in his learning rule.

We will investigate how the labour supply, the real interest rate and welfare evolve during the learning transition to a steady state. We show that if the economy initially is above (below) the steady state, then the labour supply decreases (increases) and converges to the steady state. Given certain assumptions on the preferences the steady state is globally. The welfare is increasing (decreasing) when the economy initially is above (below) the steady state.

We make a welfare comparison between a slow learner and a fast learner, born at the same time, during the entire learning transition to the steady state. We show that the welfare of a slow agent is higher (lower) than the welfare of a fast agent initially and for some time on, if the economy initially is above (below) the steady state. When the economy is "close" to the steady state the slow learner is worse (better) off than the fast learner, if the economy initially is above (below) the steady state. In fact simulations show that there are a single crossing point between the welfare sequence of the slow learners and the welfare sequence of the fast learners.

In the last part of the chapter we study three types of policies, government spending financed by seignorage, government spending financed by a lump-sum tax and a production subsidy financed by a lump-sum tax. In all three cases the welfare decreases, when we move from a steady state without any fiscal policy to a steady state where we have introduced one of the three policies. The analysis made in the previous sections covers the different cases, and since we are going to study government intervention in chapter 3 and 4, we introduce the different policies in this simple model in order to study the effects on labour, price and welfare, although government intervention is purely distortionary in this model.

In the first case, where the government finances spending by seignorage, the steady state is shifted downwards, and we can use the analysis where the economy initially are above the steady state. Hence the slow learners are initially better off than the fast learners, but eventually the fast learners become better off and remain better off for the rest of the transition phase. The welfare result is reversed when the government is using one of the other types of policy, because the steady state is shifted upward, and initially the economy are below the steady state. Thus the fast learners are initially better off, but the slow learners become and remain better off during the rest of the transition phase. In all three cases the fast agents makes a better forecast, i.e. they are "closer" to the steady state, but they need not have the highest level of welfare. There is an learning "effect" in this simple model, and in more complicated models we might expect a similar effect, when we study comparisons of different learning rules.

The chapter is organised as follows. In section 2.2, we describe the model and in section 2.3 we look at stability of the steady state under learning. Section 2.4 and 2.5

describes the evolution of the real interest rate and the welfare during the learning transition. In section 2.6, we define the fast and slow learners and look at the welfare comparison between the two economies. The effects of the various government policies are analysed in section 2.7. Section 2.8 contains the conclusion and some remarks on how to extend the results.

### 2.2. The overlapping generations model.

We consider a standard overlapping generations model with money, where an agent is born at each time t and lives for two periods. The agent works when he is young and consumes when he is old. There is one perishable good which is produced and consumed. The welfare function for an agent born at time t is given by:

$$W = U(c_{t+1}) - V(n_t),$$

where $c_{t+1}$ is consumption at time t + 1 and $n_t$ the labour supply at time t. U is the utility function and V is the labour function. We implicit assume that W is separable in c, n. In this situation the agent's problem is given by:

$$\max_{c_{t+1}, n_t, M_t^d} \quad W = U(c_{t+1}) - V(n_t)$$

subject to :

$$p_t y_t = M_t^d, \tag{2.1}$$

$$p_{t+1}^e c_{t+1} = M_t^d. \tag{2.2}$$

Where $p_{t+1}^e$ is the expected price at time t + 1, and the agents have point expectations. $y_t$ is the output produced at time t, and since labour is the only output in the production function, we have that $y_t = f(n_t)$ for all t. We have the standard assumption on U, V and f.

**Assumption 2.1.** f is a $C^2$-function for all n > 0 with f' > 0 and f" ≤ 0, U is a $C^2$-function for all c > 0 with U' > 0 and U" < 0 and V is a $C^2$-function for all n > 0 with V' > 0 and V" > 0. Furthermore, $U'(c) \to \infty$ when $c \to 0$ and $V'(n) \to 0$ when $n \to \infty$.

There are no externalities in the production function, for example, increasing returns, but in chapter 3 we will extend the model and incorporate increasing returns. The first-order condition for the agent's problem is:

$$(2.3) \qquad U'\left(\frac{p_t}{p^e_{t+1}} f(n_t)\right) \frac{p_t}{p^e_{t+1}} f'(n_t) - V'(n_t) = 0 \qquad \Rightarrow$$

$$(2.4) \qquad U'\left(\frac{p_t}{p^e_{t+1}} f(n_t)\right) \frac{p_t}{p^e_{t+1}} = V'(n_t) \frac{1}{f'(n_t)} .$$

The second-order condition for a maximum is satisfied given assumption 2.1 on U, V and f. In an equilibrium we have that the supply of money is equal to the demand of money:

$$(2.5) \qquad M^s_t = M^d_t \qquad \text{for all t.}$$

Let us assume that the money supply is constant $M^s_t = M$ for all t. If we combine the households budget constraint (2.1) with (2.5), we have :

$$(2.6) \qquad p_t y_t = p_{t-1} y_{t-1} \qquad \text{for all t.}$$

Since $y_t = f(n_t)$ for all t, we have

$$(2.7) \qquad \frac{p_{t-1}}{p_t} = \frac{f(n_t)}{f(n_{t-1})} \qquad \text{for all t.}$$

(2.7) is the real interest rate at time t, or the real wage at time t.

Let us briefly look at the case where the agent has perfect foresight such that $p^e_{t+1} = p_{t+1}$, hence $p_t / p^e_{t+1} = p_t/p_{t+1}$ and

$$(2.8) \qquad p_t/p_{t+1} = f(n_{t+1})/f(n_t),$$

We can substitute (2.8) into the first order condition (2.4), and we have *the offer curve* which describe the evolution of the labour supply over time :

$$(2.9) \qquad U'\{f(n_{t+1})\}f(n_{t+1}) = V'(n_t)f(n_t)/f'(n_t).$$

Since $V'' > 0$ and $f'' \leq 0$ the derivative of the right-hand side of (2.9) w.r.t. $n_t$ is positive :

$$\partial( V'(n)f(n)/f'(n) )/\partial n =$$

$$(2.10) \qquad \{ (V''(n)f(n) + V'(n)f'(n))f'(n) - f''(n)V'(n)f(n) \}/(f'(n))^2 > 0$$

In this case, (2.9) can be rearranged such that $n_t$ is a function of $n_{t+1}$ :

$$(2.11) \qquad n_t = F(n_{t+1}) \qquad \text{for all t.}$$

Since U, V and f are $C^2$-functions, then F is a $C^1$-function for all $n_{t+1} > 0$. F can be increasing, decreasing or hump-shaped, and this depends on the preferences, see figure 2.1a and 2.1b.

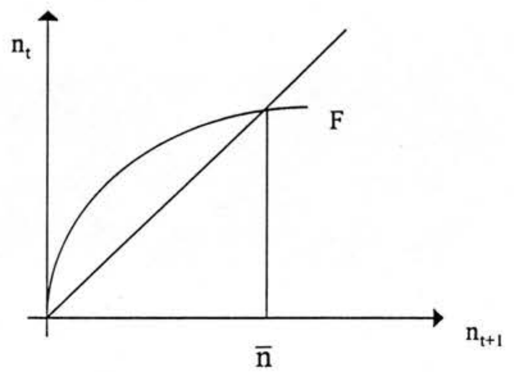


**Figure 2.1a. F increasing.**

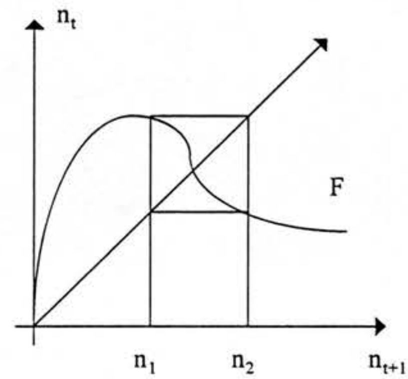


**Figure 2.1b. F hump-shaped.**

Intuitively, the shape of F depends on income- and substitution-effects. In figure 2.1a the substitution-effect dominates the income-effect, and in figure 2.1b the income-effect dominates the substitution-effect over the range where F is decreasing. The steady state $\bar{n}$ is determined by $U'(\bar{n}) = V'(\bar{n})$ so that $\bar{n}$ is a fixpoint for F, $\bar{n} = F(\bar{n})$ . Assumption 2.1 on U, V and f ensures the existence and uniqueness of an interior rational steady state $\bar{n} > 0$, see e.g. Woodford (1990). $\bar{n}$ is the well known steady state were money has value. It is possible to show the existence of a sunspot equilibrium in this model, see Woodford (1990). If furthermore $U'(c)c \to 0$ when $c \to 0$, the autarchy steady state $\bar{n} = 0$ also exists.

If F is hump-shaped it is possible that there exist cycles, an example of a 2-cycle is shown in figure 2.1b, but the following assumption on U exclude the possibility of cycles as shown by Grandmont (1985).

**Assumption 2.2.** *Grandmont (1985).* $\lambda(x) = -\dfrac{U''(x)x}{U'(x)} < 1.$

Given assumption 2.2, U'(c)c is increasing in c because

$$\partial( U'(c)c )/\partial c = U''(c)c + U'(c) > 0 \text{ if and only if}$$

$$-(U''(c)c)/U'(c) < 1$$

If the utility function U satisfies assumption 2.2, then the F-function is increasing for all n > 0. To see this, return to equation (2.9) and let the right-hand side of (2.9) be denoted by the function H : $V'(n_t)f(n_t)/f'(n_t) = H(n_t)$, then (2.9) is given by:

$$U'(f(n_{t+1}))f(n_{t+1}) = H(n_t).$$

The function H is increasing as shown in (2.10), hence H have an inverse such that :

$$n_t = H^{-1}(U'(f(n_{t+1}))f(n_{t+1})) = F(n_{t+1})$$

F is a $C^1$-function due to assumption 2.1 and the derivative of F is given by

$$F'(n_{t+1}) = \{1/H'( U'(f(n_{t+1}))f(n_{t+1}) )\}\{ U''(f(n_{t+1}))f'(n_{t+1})f(n_{t+1})$$
$$+ U'(f(n_{t+1}))f'(n_{t+1}) \} > 0$$

The derivative of F is positive, because H' > 0 and $\{U''(f(n_{t+1}))f(n_{t+1}) + U'(f(n_{t+1}))\}$ > 0 according to assumption 2.2.

**2.3. The stability of the steady state under learning.**

Let us now assume that the agent does not have perfect foresight. Thus the agent has to make a forecast of the price. However, he could also make his expectations on the output $y_{t+1}$ according to $p_t/p^e_{t+1} = y^e_{t+1}/y_t$ or forecast on $n_{t+1}$ since $y^e_{t+1} = f(n^e_{t+1})$, where $n^e_{t+1}$ is the expected labour supply at time t + 1. The expected real interest rate $p_t/p^e_{t+1}$ is given by:

(2.12) $$\frac{p_t}{p^e_{t+1}} = \frac{f(n^e_{t+1})}{f(n_t)} \qquad \text{for all t.}$$

Insert (2.12) in the first-order condition (2.4), and we have the temporary equilibrium condition:

(2.13) $$U'(f(n^e_{t+1}))\, f(n^e_{t+1}) = V'(n_t)\frac{f(n_t)}{f'(n_t)} \qquad \text{for all t.}$$

This can be rearranged as (2.11), such that the actual labour supply $n_t$ depends on the expected labour supply $n^e_{t+1}$ :

(2.14) $$n_t = F(n^e_{t+1}) \qquad \text{for all t.}$$

In this chapter the agent use the labour supply to forecast on and the forecast is made according to the adaptive learning rule given by

(2.15) $$n^e_{t+1} = n^e_t + a_t(n_{t-1} - n^e_t) = n^e_t + a_t(F(n^e_t) - n^e_t)$$

where we assume that $0 < a_t \leq 1$ for all t, and $\sum_{t=2}^{\infty} a_t = \infty$.

The agent at time t takes account of the difference made by the agent at time t – 1: $n_{t-1} - n^e_t$. As Evans and Honkapohja (1995b) state *" The agents act as if the economy was in a steady state with an unknown value, which they estimate from past data using a weighted mean of previous values* $n_s$, *s = 2,....,t – 1 ":*

$$n^e_{t+1} = a_t n_{t-1} + (1 - a_t)a_{t-1} n_{t-2} + (1 - a_t)(1 - a_{t-1})a_{t-2} n_{t-3} .....$$

It would be more natural to use the difference between the actual labour supply at time t and expected labour supply at time t : $n_t - n^e_t$, but then we would have to determine $n_t$ and $n^e_{t+1}$ simultaneously since $n_t = F(n^e_{t+1})$. Another reason might be a delay in the publication of the actual labour at time t, such that only information on $n_{t-1}$ is available at time t.

The adaptive learning rule has the merit of simplicity and there has been a wide interest in simple learning rules in the literature, see e.g. Bray (1982), Marcet and Sargent (1989a), Guesnerie and Woodford (1991) and several papers by Evans and Honkapohja. The agents are boundedly rational when they use this learning rule since they are outside a rational expectations equilibrium, but in the end the economy converges to the rational expectations equilibrium as shown below. Hence the adaptive learning rule is "reasonable" as mentioned by Bray (1982). In the paper by Bray and Savin (1986) the speed of convergence are studied. The convergence to a rational expectations equilibrium may be fast under the learning scheme as shown in Bray and Savin. Hence although agents are misspecified it is only for a short period of time before they are in a rational expectations equilibrium. In the set-up presented here simulations even fairly low values of $a_t$ generates a fast convergence to the rational expectations equilibrium, when the rational expectations equilibrium is stable under learning.

As mentioned above, the agent has a choice of which variable to use in his learning rule for example, the agent could learn on:

**(a)** The first-order condition : $x^e_{t+1} = \{U'(f(n_{t+1}))f(n_{t+1})\}^e$.

**(b)** The expected real interest rate : $x^e_{t+1} = \{p_t/p_{t+1}\}^e$.

**(c)** The labour supply : $x^e_{t+1} = n^e_{t+1}$.

We could choose (a) and let the agent learn on this "variable", but this is not a very obvious variable to learn on. It is worthwhile to look at (b), since it would be natural to make a forecast of the expected real interest rate. There is not a one-to-one correspondence between the speed of learning $\alpha_t$ in a learning rule, where the agent forecast the real interest rate, and the speed of learning $a_t$ from the learning rule (2.15). This is due to (2.15) being non-linear.

In order to simplify the set-up and make simulations, we study the following example where the utility function U and the labour function V are C.E.S.-functions and the production function f is a Cobb-Douglas function.

**Example 2.1.** *U and V are C.E.S.-functions and f is a Cobb-Douglas function.*
Let $U(c) = (1/(1-\sigma))c^{1-\sigma}$, $V(n) = (1/(1+\varepsilon))n^{1+\varepsilon}$, and $f(n) = An^{\alpha}$, where $0 < \sigma < 1$, $\varepsilon > 0$, $0 < \alpha < 1$, and $A > 0$. The function F defined in (2.14), can be written as:

(2.16) $$n_t = \alpha^{\frac{1}{1+\varepsilon}}\left[A(n^e_{t+1})^{\alpha}\right]^{\frac{1-\sigma}{1+\varepsilon}} = F(n^e_{t+1}) \qquad \text{for all t.}$$

F is a strictly concave and smooth function with two fixpoints $n_t = 0$ and $n_t = \bar{n} > 0$. The interior fixpoint $\bar{n}$ is given by $\bar{n} = B^{1/(1-\beta)}$, where $B = \alpha^{\frac{1}{1+\varepsilon}}[A]^{\frac{1-\sigma}{1+\varepsilon}} > 0$ and $\beta = \alpha(1-\sigma/(1+\varepsilon))$, such that $0 < \beta < 1$. □

Let us assume that assumptions 2.1-2.2 are satisfied, then F can be illustrated as in the following figure.

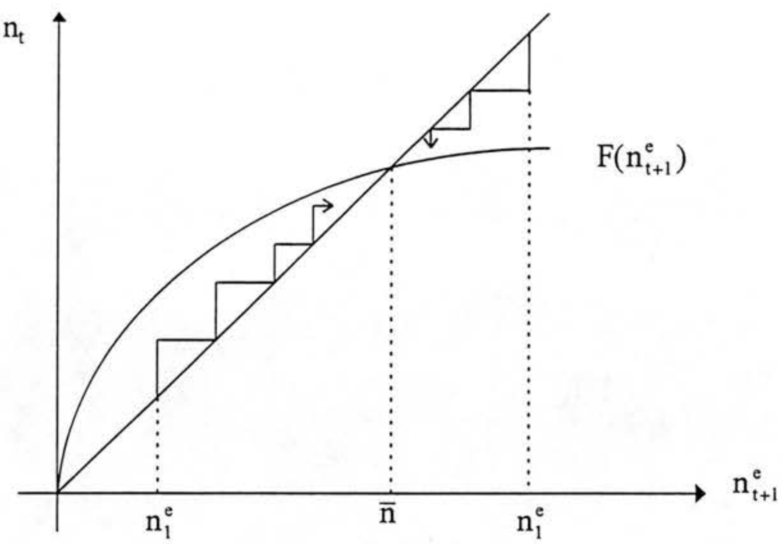


**Figure 2.2.** The F-function and the transitions paths during a learning process, that begins either below or above the steady state.

Let us analyse what happens during the learning transition to the steady state $\bar{n}$. If the initial forecast $n_1^e$ is below the steady state $\bar{n}$, then the expected labour supply increases as shown in figure 2.2. If the initial forecast $n_1^e$ is above the steady state $\bar{n}$, then the expected labour supply decreases as shown in figure 2.2. Hence we split the stability analysis into two parts, in the first case we are initially above $\bar{n}$ and in the second case we are initially below the steady state $\bar{n}$.

In the first case the initial forecast $n_1^e$ is above $\bar{n}$ :

$$n_1^e > \bar{n}, \text{ when time } t = 0.$$

Then the actual labour supply at time t = 0 is also above $\bar{n}$, since F is monotonically increasing and

$$n_0 = F(n_1^e) > F(\bar{n}) = \bar{n}.$$

At time t = 1, the expected labour supply and actual labour supply is given by :

$$n_2^e = n_1^e + a_1(n_0 - n_1^e) < n_1^e$$

since $n_0 = F(n_1^e) < n_1^e$, and the actual labour supply at time t has decreased

$$n_1 = F(n_2^e) < F(n_1^e) = n_0.$$

When $t \geq 1$, the expected labour supply is given by :

$$n^e_{t+1} = n^e_t + a_t(n_{t-1} - n^e_t) = n^e_t + a_t(\ F(n^e_t) - n^e_t\ ) \qquad \text{and}$$

$$n_t = F(n^e_{t+1}) \qquad \text{for all } t \geq 0.$$

Since $n^e_t > n_{t-1} = F(n^e_t)$ and $a_t > 0$, the expected labour supply are decreasing :

$$n^e_{t+1} < n^e_t \qquad \text{for all } t \geq 1.$$

Hence $n^e_t$ is a decreasing sequence bounded below by $\bar{n}$. Since F is continuous and monotone the actual labour supply $n_t$ is also decreasing and bounded below by the steady state $\bar{n}$.

In the second case we are initially below the steady state, $n^e_1 < \bar{n}$, and the expected labour supply, $n^e_{t+1}$, is increasing and bounded above by the steady state $\bar{n}$, as shown by figure 2.2.

Let us return to the stability question of the steady state under learning. Since we have a non-linear difference equation we are restricted to local stability, if we do not have other assumptions on U, V and f than assumption 2.1. In this case the unique interior steady state $\bar{n}$ is locally stable under learning, if

$$F'(\bar{n}) < 1,$$

as shown in Azariadis (1993) or Evans and Honkapohja (1995a). However, if F is monotonically increasing according to assumption 2.2, we can show that the steady state $\bar{n}$ is globally stable under learning.

**Proposition 2.1.** *Given assumption 2.1 - 2.2 and $F'(\bar{n}) < 1$, then the unique interior steady state is globally stable :*

$$n^e_{t+1} \rightarrow \bar{n} \text{ when } t \rightarrow \infty, \text{ for all initial values of } n^e_1.$$

**Proof.** From assumption 2.1 and 2.2, F is a continuous, monotonically increasing function with $\bar{n}$ as the unique interior steady state and we have assumed that $F'(\bar{n}) <$ 1. If $F'(\bar{n}) \geq 1$, then the steady state would be unstable.

Let $n_1^e > \bar{n}$ then $\{n_{t+1}^e\}_{t=0}^{\infty}$ is a decreasing sequence bounded below by $\bar{n}$, as shown above and illustrated in figure 2.2. Since $n_{t+1}^e$ is decreasing and bounded below by $\bar{n}$, $n_{t+1}^e$ converge to $\tilde{n} \geq \bar{n}$ when t goes to infinity. Let us assume that $\tilde{n} > \bar{n}$ and show that we obtain a contradiction.

Since F is continuous, then $F(n_{t+1}^e) \rightarrow F(\tilde{n})$ when t goes to infinity. Let $\varepsilon = F(\tilde{n}) - \tilde{n}$. From the monotonicity of F we have $F(n_t^e) - n_t^e \leq \varepsilon < 0$ for t sufficient large, and $n_{t+1}^e \leq n_t^e + a_t \varepsilon$ for t sufficiently large, such that for all $\tau \geq 1$ :

$$n_{t+\tau}^e \leq n_t^e + \varepsilon \sum_{i=t}^{t+\tau} a_i \ .$$

If we then choose $\tau$ sufficiently large and use $\sum_{t=2}^{\infty} a_t = \infty$, we have a contradiction. Thus $n_{t+1}^e \rightarrow \bar{n}$ when $t \rightarrow \infty$ . The argument for convergence when $n_1^e < \bar{n}$ is analogous. ■

Since F is a continuous function, $n_t$ converges towards $\bar{n}$ when t goes to infinity. We can illustrate the paths for expected labour supply $n_{t+1}^e$ and actual labour supply $n_t$, when the economy initially is above the steady state or initially is below the steady state.

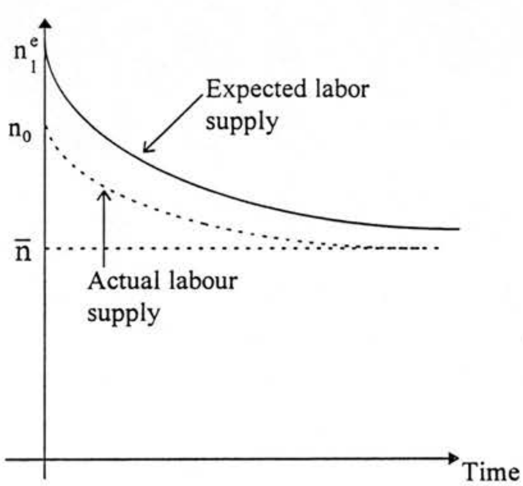


**Figure 2.3a.** $n_1^e > \bar{n}$,

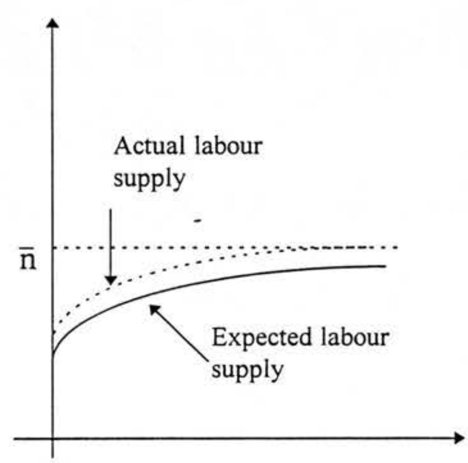


**Figure 2.3b.** $n_1^e < \bar{n}$,

## 2.4. The real interest rate during the learning transition.

Here we will again separate the analysis into two cases, either the economy is above the steady state initially or the economy is below the steady state initially. The expected real interest rate at time t is given by :

$$(2.17) \qquad \frac{p_t}{p^e_{t+1}} = \frac{f(n^e_{t+1})}{f(n_t)} \qquad \text{for all t.}$$

The actual real interest rate at time t is given by :

$$(2.18) \qquad \frac{p_{t-1}}{p_t} = \frac{f(n_t)}{f(n_{t-1})}.$$

Let us study the paths of the expected and actual real interest rates when the economy is either above or below the steady state initially. At the steady state the real interest rate is equal to 1, $p_t/p_{t-1} = 1$.

In the first case, where the economy is initially above the steady state, $n^e_1 > \bar{n}$, we have that $n_0 < n^e_1$ and $p_0/p^e_1 = f(n^e_1)/f(n_0) > 1$. Since $n_t < n^e_{t+1}$ for all $t \geq 1$ then $f(n_t) < f(n^e_{t+1})$ for all $t \geq 1$, and from (2.17) the expected real interest rate is larger than 1 :

$$p_t/p^e_{t+1} > 1 \qquad \text{for all t.}$$

Since $n_t$ is a decreasing sequence in this case the actual real interest rate is less than 1 according to (2.18)

$$p_t/p_{t+1} < 1 \qquad \text{for all t.}$$

In the second case where $n^e_1 < \bar{n}$, thus $n_0 > n^e_1$ and $p_0/p^e_1 = f(n^e_1)/f(n_0) < 1$. Since $n_t > n^e_{t+1}$ for all $t \geq 1$ then $f(n_t) > f(n^e_{t+1})$ for all $t \geq 1$, and

$$p_t/p^e_{t+1} < 1 \qquad \text{for all t.}$$

Since $n_t$ is a increasing sequence in this case, then

$$p_t/p_{t+1} > 1 \qquad \text{for all t.}$$

In both cases, the expected and actual real interest rates converge to the steady state level :

$$p_t / p^e_{t+1} \to 1 \text{ for } t \to \infty,$$

$$p_t / p_{t+1} \to 1 \text{ for } t \to \infty,$$

because f and F are continuous functions. In proposition 2.2, we will show that the expected real interest rate $p_t / p^e_{t+1}$ decreases (increases) monotonically and the actual real interest rate $p_t/p_{t+1}$ increases (decreases) monotonically, when we initially begin above (below) the steady state.

**Proposition 2.2.** *(a) If the economy initially is above the steady state, $n^e_1 > \bar{n}$, then the expected real interest rate is decreasing : $p_t / p^e_{t+1} > p_{t+1} / p^e_{t+2}$ .*
*If U and V are as in example 2.1, then the actual real interest rate is increasing : $p_t / p_{t+1} < p_{t+1}/p_{t+2}$ .*
*(b) If the economy initially is below the steady state, $n^e_1 < \bar{n}$, then the expected real interest rate is increasing : $p_t / p^e_{t+1} < p_{t+1} / p^e_{t+2}$.*
*If U and V are as in example 2.1, then the actual real interest rate is decreasing : $p_t / p_{t+1} > p_{t+1}/p_{t+2}$.*

**Proof.** (a) If we return to the first-order condition, then we can write the expected real interest rate $p_t / p^e_{t+1}$ as a function of $n_t$ . Set $p_t / p^e_{t+1} = r^e_{t+1}$ then the first-order condition is given by:

$$U'(r^e_{t+1} f(n_t))\, r^e_{t+1} = V'(n_t)/f(n_t) \qquad \Rightarrow$$

$$n_t = h(r^e_{t+1}),$$

where the derivative of h w.r.t. $r^e_{t+1}$ is positive since

$$\partial\{U'(r^e_{t+1} f(n_t))\, r^e_{t+1}\}/\partial r^e_{t+1} = U''(r^e_{t+1} f(n_t))\, r^e_{t+1} + U'(r^e_{t+1} f(n_t)) > 0$$

according to assumption 2.2 and $h'(r^e_{t+1}) > 0$. Since $n_t$ is decreasing when the economy initially above the steady state, then $r^e_{t+1}$ is decreasing as well.

Let us assume that U, V and f are as in example 2.1. We can write the actual interest rate as a function of $n^e_{t+1}$ :

$$\frac{p_t}{p_{t+1}} = \frac{f(n_{t+1})}{f(n_t)} = \frac{f(F(n^e_{t+2}))}{f(F(n^e_{t+1}))} = \frac{f\{F(n^e_{t+1} + a_t(F(n^e_{t+1}) - n^e_{t+1}))\}}{f(F(n^e_{t+1}))} = \xi(n^e_{t+1}).$$

Since $n^e_{t+1} > n^e_{t+2}$ then $p_t/p_{t+1} < p_{t+1}/p_{t+2}$ if $\xi'(n^e_{t+1}) < 0$. Given the expressions for f and F : $f(n) = An^\alpha$ and $F(n) = Bn^\beta$, it is shown in the appendix to chapter 2 that $\xi'(n^e_{t+1}) < 0$. Hence the actual interest rate is increasing.

(b) Since $h'(r^e_{t+1}) > 0$ and $n^e_{t+1}$ is increasing then the expected real interest rate is increasing. Furthermore $\xi'(n^e_{t+1}) < 0$ and $n^e_{t+1}$ is increasing such that the actual real interest rate is decreasing. ■

Proposition 2.3 is illustrated in the following figure.

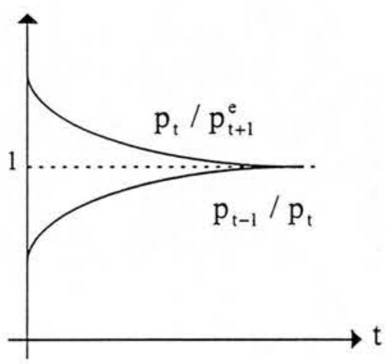


**Figure 2.4a**. The expected and actual real interest rate, when $n^e_1 > \bar{n}$.

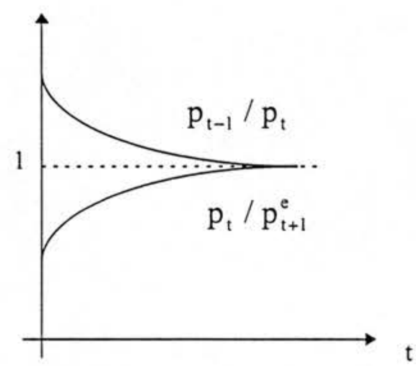


**Figure 2.4b**. The expected and actual real interest rate, when $n^e_1 < \bar{n}$.

The difference between the expected and actual real interest rates are due to the agents using the learning rule such that they use a misspecified model. If the speed of learning is high the convergence to the steady state value of the real interest rate is high, and they will only use a "wrong" for a short number periods.

### 2.5. The welfare path during the learning transition.

The welfare for an agent born at time t is given by: $U(c_{t+1}) - V(n_t)$. In a temporary equilibrium we have $n_t$ from (2.14):

$$n_t = F(n^e_{t+1}) \qquad \text{for all } t \geq 0.$$

The consumption $c_{t+1}$ in an equilibrium depends on the actual output in period t + 1, thus the actual consumption is given by:

$$\begin{aligned} c_{t+1} &= f(n_{t+1}) = f(F(n^e_{t+2})) \\ &= f(F\{n^e_{t+1} + a_t(F(n^e_{t+1}) - n^e_{t+1})\}) \qquad \text{for all } t \geq 0. \end{aligned}$$

This can be substituted into the private welfare function, and we can study how the private welfare depends on the learning rule.

**Proposition 2.3.** *Given assumption 2.1 and 2.2. If the economy initially is above the steady state, the welfare in the steady state* $(\bar{n}, \bar{c})$ *is greater than the welfare for an agent born at time t, where* $(n_t, c_{t+1})$ *denotes the agent's actual labour supply and consumption in a temporary equilibrium:*

*(i)* $\qquad U(c_{t+1}) - V(n_t) \leq U(\bar{c}) - V(\bar{n}), \quad \text{for all } t \geq 0.$

*The private welfare sequence* $\{U(c_{t+1}) - V(n_t)\}_{t=0}^{\infty}$ *is convergent:*

*(ii)* $\qquad U(c_{t+1}) - V(n_t) \rightarrow U(\bar{c}) - V(\bar{n}), \text{ when } t \rightarrow \infty.$

*Furthermore, if U, V and f are as in example 2.1 and assume that* $a_t = a$ *for all t, then welfare is monotonically increasing :*

*(iii)* $\qquad U(c_{t+1}) - V(n_t) < U(c_{t+2}) - V(n_{t+1}).$

*If the economy initially is below the steady state, then the welfare in the steady state* $(\bar{n}, \bar{c})$ *is less than the welfare for an agent born at time t, where* $(n_t, c_{t+1})$ *denotes the agent's labour supply and consumption in a temporary equilibrium:*

*(iv)* $\qquad U(c_{t+1}) - V(n_t) > U(\bar{c}) - V(\bar{n}), \quad \text{for all } t \geq 0.$

*The welfare sequence* $\{U(c_{t+1}) - V(n_t)\}_{t=0}^{\infty}$ *is convergent:*

*(v)* $\qquad U(c_{t+1}) - V(n_t) \rightarrow U(\bar{c}) - V(\bar{n}), \text{ when } t \rightarrow \infty.$

*Furthermore, if U, V and f are as in example 2.1 and assume that* $a_t = a$ *for all t, then welfare is monotonically decreasing :*

*(vi)* $\qquad U(c_{t+1}) - V(n_t) > U(c_{t+2}) - V(n_{t+1}).$

**Proof.** *(i)* Since $n_t > \bar{n}$ for all $t \geq 0$, we have to prove :

$$V(n_t) - V(\bar{n}) > U(c_{t+1}) - U(\bar{c}).$$

From assumption 2.1, we have that V is strictly convex, hence

$$V(n_t) - V(\bar{n}) > V'(\bar{n})(n_t - \bar{n}), \tag{2.19}$$

because $n_t > \bar{n}$ for all $t \geq 0$. When $n_t = \bar{n}$ we have from (2.9) that

$$V'(\bar{n}) = U'(\bar{c})f'(\bar{n}).$$

Thus the right-hand side of (2.19) is equal to

$$= U'(\bar{c})f'(\bar{n})(n_t - \bar{n})$$

Since U is strictly concave and $c_{t+1} > \bar{c}$ for all t, we have:

$$U'(\bar{c})f'(\bar{n})(n_t - \bar{n}) > \left(\frac{U(c_{t+1}) - U(\bar{c})}{c_{t+1} - \bar{c}}\right) f'(\bar{n})(n_t - \bar{n})$$

$$= \left(U(c_{t+1}) - U(\bar{c})\right)\left(\frac{f'(\bar{n})(n_t - \bar{n})}{f(n_{t+1}) - f(\bar{n})}\right) \tag{2.20}$$

Since $n_t > n_{t+1} > \bar{n}$ and f is concave (2.20) is larger than

$$> \left(U(c_{t+1}) - U(\bar{c})\right)\left(\frac{f'(\bar{n})(n_{t+1} - \bar{n})}{f(n_{t+1}) - f(\bar{n})}\right) \tag{2.21}$$

Since f is concave and $n_t > \bar{n}$, then (2.21) is larger than

$$> U(c_{t+1}) - U(\bar{c}).$$

Hence the left-hand side of (2.19) is larger than $U(c_{t+1}) - U(\bar{c})$ and we the desired result.

*(ii)* Since $n_t$ and $c_{t+1}$ converges to $\bar{n}$ and $\bar{c}$, and U and V are continuous functions, we have :

$$U(c_{t+1}) - V(n_t) \rightarrow U(\bar{c}) - V(\bar{n}), \text{ when } t \rightarrow \infty.$$

*(iii)* We can write the private welfare as a function of the expected labour supply in period t + 1:

$$U(c_{t+1}) - V(n_t) = U(f(n_{t+1})) - V(F(n^e_{t+1}))$$

$$= U(f\{F(n^e_{t+2})\}) - V(F(n^e_{t+1}))$$

$$= U(f(F[n^e_{t+1} + a(F(n^e_{t+1}) - n^e_{t+1})])) - V(F(n^e_{t+1}))$$

$$= \omega(n^e_{t+1}, a).$$

If $\omega(n^e_{t+1},a) < \omega(n^e_{t+2},a)$ then $U(c_{t+1}) - V(n_t) < U(c_{t+2}) - V(n_{t+1})$. Since $n^e_{t+1} > n^e_{t+2}$, it is sufficient to prove that $\omega(\,.\,,a)$ is a decreasing function for:

$$\frac{\partial\omega(n^e_{t+1},a)}{\partial n^e_{t+1}} < 0.$$

With the expressions for U, V and f given in example 2.1 and the equilibrium condition (2.9) we can determine $\partial\omega(n^e_{t+1},a)/\partial n^e_{t+1}$, and show that it is less than 0.

$$(2.22)\quad \frac{\partial\omega(n^e_{t+1},a)}{\partial n^e_{t+1}} = U'(c_{t+1})f'(n_{t+1})F'(n^e_{t+2})\frac{\partial\psi(n^e_{t+1},a)}{\partial n^e_{t+1}} - V'(n_t)F'(n^e_{t+1})$$

where $\psi(n^e_{t+1},a) = n^e_{t+1} + a(F(n^e_{t+1}) - n^e_{t+1})$. Insert the equilibrium condition (2.9) into (2.22) and we have:

$$(2.23)\qquad = V'(n_t)\frac{f(n_t)}{f(n_{t+1})}\frac{1}{f'(n_t)}f'(n_{t+1})F'(n^e_{t+2})\frac{\partial\psi(n^e_{t+1},a)}{\partial n^e_{t+1}} - V'(n_t)F'(n^e_{t+1}).$$

Insert the expressions for U, V, f and F given in example 2.1, then the calculations in the appendix to chapter 2 show that (2.23) is negative, and $\partial\omega(n^e_{t+1},a)/\partial n^e_{t+1}$ is negative.

*(iv) and (v)* This is proven similarly to the proof of *(i)* and *(ii)*.

*(vi)* In the proof of *(iii)* we showed that $\partial\omega(n^e_{t+1},a)/\partial n^e_{t+1} < 0$. Since $n^e_{t+1}$ is increasing then the welfare must be decreasing during the learning transition. ■

The welfare will thus be increasing (decreasing) if the economy initially are above (below) the steady state during the learning transition.

### 2.6. An economy of fast learners compared with an economy of slow learners.

We will now make a comparison between an economy of fast learners and an economy of slow learners. We will show that the welfare of the fast learner is less (larger) than the welfare of the slow learner initially, and when t is sufficiently large the fast learners are better (worse) off than the slow learners, if $n_1^{f,e} = n_1^{s,e} > \bar{n}$ ($n_1^{f,e} = n_1^{s,e} < \bar{n}$). Let us define what we mean by fast and slow learners. Let " f " denote the fast learner and " s " denote the slow learner.

**Definition 2.1.** Let $a_t^f$ denote the weight in the learning rule (2.15) for one group of agents, and let $a_t^s$ denote the weight in (2.15) for another group of agents such that $a_t^f$ and $a_t^s$ satisfies the following condition :

$$a_t^f > a_t^s \qquad \text{for all t.}$$

The group with $a_t^f$ are called fast learners, while the group with $a_t^s$ are called slow learners.

The difference between the two groups of learners is only in the speed of learning and it is not possible to change the speed of learning within a group, such that $a_t^f < a_t^s$ for some t. The terminology slow and fast learner is not meant as the fast agents are learning more than the slow learners at each point in time, and suddenly end up with perfect foresight. A fast learner have a quicker response in his learning rule compared to the slow learner, but both types of agents are using the learning rule and do not know the true value of the variable they forecast.

**Example 2.2.** *The economy are initially above the steady state.* Let U,V and f be the C.E.S.-functions from example 2.1 with the following parameter values : A = 28, $\varepsilon$ = 0.2, $\sigma$ = 0.41 and $\alpha$ = 0.2. The speed of learning is given by $a_t^s = a_s = 0.2$ for all t and $a_t^f = a_f = 0.5$ for all t.

We compare the two separate economies, given the initial condition

(2.24) $\quad n_1^{s,e} = n_1^{f,e} > \bar{n}$.

The first economy consists of only slow learners, where all agents use $a_t^s = 0.2$ in their learning rule. We calculate the expected labour supply at time t + 1 denoted by $n_{t+1}^{s,e}$ given the initial condition (2.24). From the expected labour supply we can calculate the actual labour supply at time t, $n_t^s$, the consumption at time t + 1, $c_{t+1}^s$, and the private welfare $U(c_{t+1}^s) - V(n_t^s)$ during the movement to the steady state $\bar{n}$.

The second economy is an economy with only fast learners. All agents use $a_t^f = 0.5$. We make the same calculations for the fast learner given the initial condition (2.24) : $n_{t+1}^{f,e}$, $n_t^f$, $c_{t+1}^f$ and $U(c_{t+1}^f) - V(n_t^f)$.

We can now compare the welfare between a fast learner born at time t and a slow learner born at time t from each of these two economies. Since the steady state is globally stable under learning, they converge to the steady state. Given the parameter values, we can plot the private welfare over time for the slow and fast learners in a figure.

Figure 2.5 about here.

Figure 2.5 is generated given the initial conditions : $30 = n_1^{s,e} = n_1^{f,e} > \bar{n} \approx 1.39$. □

**Example 2.3.** *The economy are initially below the steady state.* Let U,V and f be the C.E.S.-functions as in example 2.1 with the following parameter values : A = 28, $\varepsilon$ = 0.2, $\sigma$ = 0.41, $\alpha$ = 0.2 and $\gamma$ = 0.2. The speed of learning is given by $a_t^s = a_s = 0.2$ for all t and $a_t^f = a_f = 0.5$ for all t.

We compare the two separate economies, given the initial condition

(2.25) $\quad n_1^{s,e} = n_1^{f,e} < \bar{n}$.

The welfare comparison between fast and slow learners are in figure 2.6 and as figure 2.6 shows, we have the opposite result to example 2.2.

Figure 2.6 about here

Figure 2.6 is generated with the initial conditions : $0.5 = n_1^{s,e} = n_1^{f,e} < \bar{n} \approx 1.39$. □

Since figure 2.6 is just opposite of figure 2.5, we concentrate on figure 2.5. The fast learner have initially a lower welfare than the slow learner, but at some time T the fast learner overtakes the slow learner and have a higher welfare for all $t \geq T$. The intuition behind this result is that the slow learners are compensated for being slow during the initial periods of the transition phase, since the generation at time 0 benefits from the generation at time 1 being are slow. The crossing point between the fast learner's and the slow learner's welfare depends on the parameter values, and it is possible to choose the parameter values such that only the slow learner born at time 0 is better off compared to the fast learners.

The fast learners eventually become better off than the slow learners, because the fast learners make a "better" forecast, i.e. they are closer to the steady state $\bar{n}$ :

$$| n_{t+1}^{f,e} - \bar{n} | < | n_{t+1}^{s,e} - \bar{n} | \qquad \text{for all } t \geq 1.$$

Furthermore since welfare is increasing over time, future generations of fast learners benefits from the previous generations being fast learners.

It seems that the single-crossing point property does not depend on specific parameter values, since different parameter values give the same picture, although the crossing point varies. However it was not possible to prove this single-crossing property, since there are two effects on welfare that works in opposite directions. In the proof of proposition 2.3, we showed that welfare depended negatively on expected labour supply :

$$\partial\omega(n^e,a)/\partial n^e < 0 \qquad \text{for all } n^e > 0,$$

where $n^e$ denotes expected labour supply. Since the expected labour supply is decreasing, the welfare is increasing. However the welfare depend negatively on the speed of learning when $\overline{n} < n^{f,e}_{t+1} < n^{s,e}_{t+1}$ :

$$\partial\omega(n^e,a)/\partial a < 0,$$

hence a low speed of learning have positive effect on welfare, when $\overline{n} < n^{f,e}_{t+1} < n^{s,e}_{t+1}$.

In proposition 2.4 below, we show a weaker result that the slow learners are better off initially, but there exists an integer $\tau$ sufficiently large, such that the fast learners are better of than the slow learners for all $t \geq \tau$. We need an extra assumption on the weights $a^f_t$ and $a^s_t$ to prove proposition 2.4.

**Assumption 2.3.** $a^s_t = a_s$ for all t, $a^f_t = a_f$ for all t.

From definition 2.1 of $a^f_t$ and $a^s_t$, we have $a_s < a_f$.The following lemma is needed for the proof of proposition 2.4.

**Lemma 2.1.** *Let $(x_t)_{t=0}^{\infty}$ and $(y_t)_{t=0}^{\infty}$ denote two sequences given by:*

$$x_{t+1} = \theta_1 x_t + \varepsilon(x_t) \quad \text{for all } t \geq 0,$$

$$y_{t+1} = \theta_2 y_t + \varepsilon(y_t) \quad \text{for all } t \geq 0,$$

*where $0 < \theta_2$, $0 < \theta_1$ and $\varepsilon$ is a continuous function with $\varepsilon(0) = 0$ and $\lim_{x\to 0} \frac{\varepsilon(x)}{x} = 0$. Furthermore, let us assume that both sequences are positive and converge to 0 when t goes to infinity :*

$$x_t > 0,\ y_t > 0 \text{ for all } t \geq 0,$$

$$x_t \to 0 \quad \text{when } t \to \infty$$

$$y_t \to 0 \quad \text{when } t \to \infty.$$

*If $\theta_2 < \theta_1$, then there exists an integer $\tau$ such that :*

$$x_t > y_t \quad \text{for all } t \geq \tau.$$

**Remark.** The reason for assuming that $x_t$ and $y_t$ converge to 0 when t goes to infinity is to avoid the possibility of $x_t$ ( and $y_t$ ) are converging to the stable point $x^* = \varepsilon(x^*)/(1-\theta_1)$ (and $y^* = \varepsilon(y^*)/(1-\theta_2)$ ). The function $\varepsilon$ could be the remainding term from a Taylor-approximation. This is how we use lemma 2.1 in proposition 2.4 and 2.5, where we linearise a non-linear difference equation around the steady state.

The lemma tells us something of the speed of convergence between $x_t$ and $y_t$ when t goes to infinity. Since $y_t$ has a lower $\theta$, then it will converge faster toward 0 compared to $x_t$ when t goes to infinity, assuming both sequences converge to 0.

**Proof of lemma 2.1.** Look at the sequence defined by $\left(\frac{y_t}{x_t}\right)_{t=0}^{\infty}$, this is well defined since $x_t > 0$ for all t and is given by:

$$\frac{y_{t+1}}{x_{t+1}} = \frac{\theta_2 y_t + \varepsilon(y_t)}{\theta_1 x_t + \varepsilon(x_t)} = \frac{\theta_2 y_t}{\theta_1 x_t}\left(\frac{1+\frac{\varepsilon(y_t)}{\theta_2 y_t}}{1+\frac{\varepsilon(x_t)}{\theta_1 x_t}}\right) = \frac{\theta_2 y_t}{\theta_1 x_t}\mu(x_t, y_t).$$

$\mu(x_t, y_t)$ is a well-defined continuous function since $x_t > 0$ and $y_t > 0$ for all t. When t is going to infinity then $x_t \to 0$ and $y_t \to 0$, and $\mu(x_t, y_t) \to 1$. Since $\theta_2 < \theta_1$, there exists an integer p such that :

$$\frac{\theta_2}{\theta_1}\mu(x_t, y_t) < r < 1, \quad \text{for all } t \geq p.$$

When t is sufficiently large, we have :

$$\frac{y_{t+1}}{x_{t+1}} < r\frac{y_t}{x_t} \quad \text{for all } t \geq m.$$

From this inequality, we have :

$$\frac{y_s}{x_s} < r^{s-m}\frac{y_m}{x_m}.$$

Since $0 < r < 1$ : $r^{s-m} \to 0$ when $s \to \infty$, such that $\frac{y_s}{x_s} \to 0$ when $s \to \infty$, and there exist an integer $\tau > m$ such that:

$$\frac{y_t}{x_t} < 1 \quad \text{for all } t \geq \tau \;\Rightarrow\; y_t < x_t \;\text{ for all } t \geq \tau. \;\blacksquare$$

Proposition 2.4 is a local result, and we assume that $F'(\bar{n}) < 1$ in the following such that $\bar{n}$ is locally stable.

**Proposition 2.4. The economy is initially above the steady state.**

*Let* $(n_t^s, c_{t+1}^s, p_t^s)_{t=0}^{\infty}$ *and* $(n_t^f, c_{t+1}^f, p_t^f)_{t=0}^{\infty}$ *denote the temporary equilibrium labour supply, consumption and price for the slow and fast learners, respectively, Given assumptions 2.1, 2.2 and 2.3. If* $n_0^{e,f} = n_0^{e,s} > \bar{n}$ *then*

*(a) The slow learner born at time* $t = 0$ *is better off than the fast learner born at time* $t = 0$ *:*

$$U(c_1^s) - V(n_0^s) > U(c_1^f) - V(n_0^f).$$

*(b) There exists an integer* $\tau$ *sufficiently large, such that the fast learners are better off than slow learners for all* $t \geq \tau$ *:*

$$U(c_{t+1}^s) - V(n_t^s) < U(c_{t+1}^f) - V(n_t^f) \quad \textit{for all } \; t \geq \tau.$$

**Proof.** The proof is divided into two parts. In part 1, we show that the slow learner is better off at time t = 0. In part 2 we show there exist a $\tau$ sufficiently large, such that the fast learner is better off and stays better off for all $t > \tau$. Both parts are shown in the appendix to chapter 2. ■

Figure 2.5 can be used as an illustration of the proposition 2.4. We have compared a group of fast learners with a group of slow learners. The slow learners benefit from everybody being slow initially, where the slow is better off than the fast. When the agents converge towards the new steady state the fast learner becomes better off, because his forecast is closer to the true optimum compared to the forecast of the slow. Hence initially the slow learners benefit from their slow learning rule, because they do not adjust their forecast as quickly as the fast learners, and the fast learner make a better forecast such that they are better off in the end.

If the economy is below the steady state as in example 2.3 then we can show the opposite result to proposition 2.4.

**Proposition 2.5. The economy is initially below the steady state.**

*Let* $(n_t^s, c_{t+1}^s, p_t^s)_{t=0}^{\infty}$ *and* $(n_t^f, c_{t+1}^f, p_t^f)_{t=0}^{\infty}$ *denote the temporary equilibrium labour supply, consumption and price for the slow and fast learners, respectively, Given assumptions 2.1, 2.2 and 2.3. If* $n_0^{e,f} = n_0^{e,s} < \bar{n}$ *then*

*(a) The fast learner born at time* $t = 0$ *is better off than the slow learner born at time* $t = 0$ *:*

$$U(c_1^s) - V(n_0^s) < U(c_1^f) - V(n_0^f).$$

*(b) There exists an integer* $\tau$ *sufficiently large, such that the slow learners are better off than fast learners for all* $t \geq \tau$*:*

$$U(c_{t+1}^s) - V(n_t^s) > U(c_{t+1}^f) - V(n_t^f) \quad \text{for all } t \geq \tau.$$

**Proof.** This is similar to the proof of proposition 2.4.

Proposition 2.5 can be illustrated by figure 2.6 above. In this case the slow learners welfare do not decrease as fast as the fast learners welfare, due to their slower adjustment. The initial generation however of fast learners are better off because the next generation of fast learners born at time are working harder than the generation of slow learners born at time 1.

### 2.7. Three different cases of government intervention.

We will look at three cases of government policies in this chapter in order to study the welfare-effects of different speeds of learning. In the first first case the government buys some of the good and finance the purchase by money-creation according to:

(2.26) $\qquad M_t^s = M_{t-1}^s + p_t g_t$ , for all t.

$M_t^s$ is the money supply at time t and $p_t$ is price at time t, and $g_t$ is government purchases at time t. In the second case, government spending $g_t$ is financed by a lump-sum tax $T_t$ and the money supply is held constant : $M_t^s = M$ for all t. The budget constraint is given by:

(2.27) $\qquad T_t = p_t g_t$ , for all t.

We assume in both cases that the government buys a fraction $\gamma_t$ of the production of the good $f(n_t)$ at time t:

(2.28) $\qquad g_t = \gamma_t f(n_t)$ , where $0 \leq \gamma_t < 1$.

In the third case we assume the government introduces a production subsidy $s_t$ in each period. This is financed by a lump-sum tax and money supply is held constant $M_t^s = M$ for all t, such that the budget constraint is given by :

(2.29) $\qquad T_t = s_t p_t f(n_t)$, for all t.

In all three cases, the model is purely deterministic, and there is no random shock in the model. In chapter 4, we extend the model to include a shock either to the preferences or as a productivity shock.

#### 2.7.1. Government spending financed by money creation.

In the first case the government introduce spending at time T, and this is unanticipated by the agents. Let us assume that until time T > 0, there are no government spending such that $\gamma_t = 0$ for all $t \leq T - 1$ and that the economy is in the steady state where $\gamma_t = 0 : n_{T-1} = \bar{n}$. At time T, $\gamma_t$ is increased and

$$\gamma_t = \gamma > 0 \qquad \text{for all } t \geq T.$$

We assume that spending is constant. The equilibrium condition on the money market is given by :

$$M_t^s = M_t^d \qquad \text{for all t,}$$

Since $M_t^d = p_t y_t$ for all t, the money market equilibrium condition combined with (2.26) give the following condition:

$$p_t y_t = p_{t-1} y_{t-1} + \gamma p_t y_t \qquad \Rightarrow$$

$$\text{(2.30)} \qquad \frac{p_{t-1}}{p_t} = (1-\gamma)\frac{f(n_t)}{f(n_{t-1})} \qquad \text{for all } t \geq T.$$

Hence if agents have perfect foresight $p_{t+1}^e = p_{t+1}$, the offer curve (2.9) is changed to

$$U'\{(1-\gamma)f(n_{t+1})\}(1-\gamma)f(n_{t+1}) = V'(n_t)f(n_t)/f'(n_t) \quad \text{for all } t \geq T.$$

Since $V'' > 0$ and $f'' \leq 0$, we write $n_t$ as a function of $n_{t+t}$ and $\gamma$:

$$n_t = F(n_{t+1}, \gamma)$$

If U satisfies assumption 2.2, then the F is rotated downwards when $\gamma$ is increased as shown in figure 2.7.

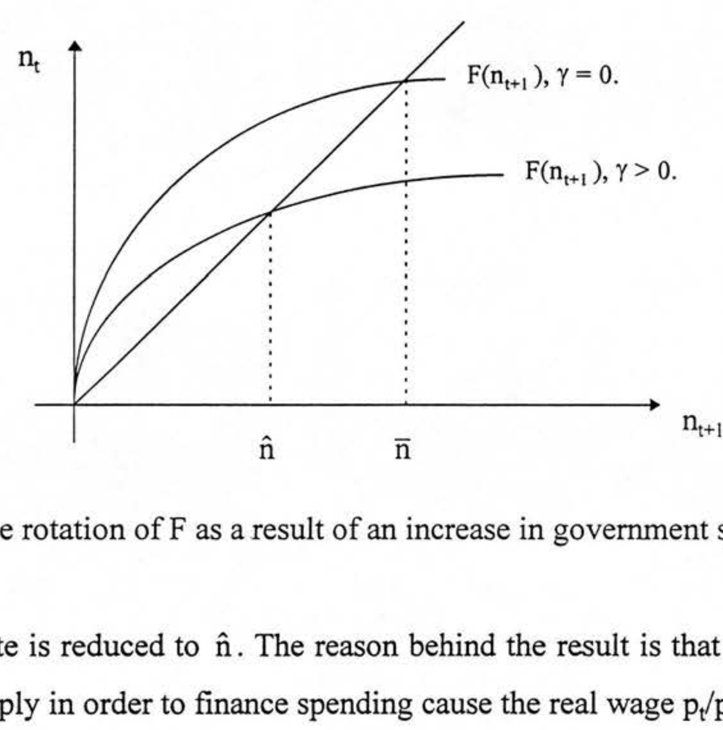


**Figure 2.7.** The rotation of F as a result of an increase in government spending.

The steady state is reduced to $\hat{n}$. The reason behind the result is that the increase in the money-supply in order to finance spending cause the real wage $p_t/p_{t+1}$ to decrease. A fall in the real wage lowers the return on work and saving, which leads to a fall in the labour-supply, since the substitution-effect dominates the income-effect.

If the agents do not have perfect foresight but use the learning rule (2.15), the temporary equilibrium condition (2.13) is given by $n_t = F(n_{t+1}^e, \gamma)$. We can use proposition 2.1 to show that the steady state $\hat{n}$ is stable under learning. The welfare comparison made in section 2.6 can be used to describe this case. The initial condition is

$$n_1^{f,e} = n_1^{s,e} = \bar{n} > \hat{n}.$$

Given this initial condition we can use proposition 2.4, and the slow learners are better off initially but when t is large and we are "close" to the new steady state the fast is better off. If we compare the welfare between the two steady states, the steady state welfare decrease after the government intervention, since the government takes away consumption from the household. During the learning transition from the old steady state to the new steady state the household are actually pushed below the new level of steady state welfare. This is a result of the government taking away consumption from the households, the households are hit by an inflation tax through (2.30), and furthermore they are not solving the right optimisation problem when they are using the learning rule. This welfare paths for the two economies are shown in figure 2.8.

Figure 2.8 about here.

Figure 2.8 illustrates the situation where the economy initially are in the steady state and remains there until $t = 50$, when $t = 50$ the government increase spending, $\gamma = 0.2$ and the parameter values from example 2.1. This change the steady state, and the agents converge to this new steady state. The last part of figure is not very clear, but go back to figure 2.5, for an illustration when $t > 50$.

### 2.7.2. Government spending financed by a lump-sum tax.

We analyse the case where the government use a lump-sum tax and keep the money supply constant $M_t^s = M$. Let us assume the government increase spending at time T

$$\gamma_t = \gamma > 0 \qquad \text{for all } t \geq T.$$

The households budget-constraint at time t is changed since it has to pay the lump-sum $T_t$:

$$p_t f(n_t) - T_t = M_t^d$$

The equilibrium condition on the money market is given by

$$M = M_t^d$$

This combined with the new budget-constraint gives :

$$p_t f(n_t) - T_t = M \qquad \Rightarrow \qquad (1-\gamma)p_t f(n_t) = M \qquad \Rightarrow$$

$$p_t / p_{t+1} = f(n_{t+1})/f(n_t) \quad \text{for all } t \geq T,$$

However at time T, then

$$p_{T-1}/p_T = (1-\gamma)f(n_T)/f(n_{T-1}) \text{ or}$$

$$p_{T-1}/((1-\gamma)p_T) = f(n_T)/f(n_{T-1})$$

The increased spending at time T works more like a subsidy to the price $p_T$.The offer curve is changed to :

$$U'\{(1-\gamma)f(n_{t+1})\}f(n_{t+1}) = V'(n_t)f(n_t)/f'(n_t) \text{ for all } t \geq T.$$

We can again write $n_t$ as a function of $n_{t+t}$ :

$$n_t = F(n_{t+1}, \gamma)$$

Since U" < 0 then F is shifted upwards when $\gamma$ is increased as shown in figure 2.9.

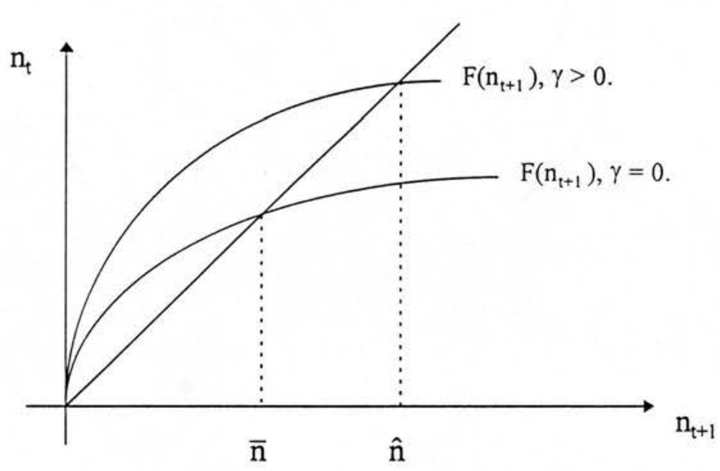


**Figure 2.9.** The upward rotation in F as a result of an increase in government spending.

In this case the real wage is increasing instead of decreasing as above, and the household receive a higher return on work. Since substitution effect still dominates the income effect the labour supply increase. The welfare comparison between the fast and slow learners are similar to proposition 2.5 where the economy initially is below the steady state, since the initial condition are :

$$n_1^{f,e} = n_1^{s,e} = \bar{n} < \hat{n}.$$

The fast learners are better off initially, but when t is large and the economy is close to the new steady state the slow learners are better off. The steady state welfare decreases although the labour supply increases. The government intervention is still distortionary. However, the households can initially have a higher welfare than the welfare at the steady state $\bar{n}$, this is due to the goverment spending works more like a subsidy to price for the old generation at time T, since the change was unanticipated by the agents at time T. Hence the old generation at time T can benefit from the introduction of government spending compared to the old steady state level $U(\bar{c}) - V(\bar{n})$. The welfare for the two economies can be illustrated by the follwing figure.

Figure 2.10 about here.

This result is again generated where the economy is in a steady state for the first 50 periods and then spending is increased at time t = 50, with $\gamma = 0.2$ and the parameter values in example 2.1.

### 2.7.3. A subsidy to the price of output financed by a lump-sum tax.

As a last case, we look at a subsidy to the price on output. This is in many respects similar to spending financed by a lump-sum tax. The government introduce a subsidy to the price of production according to (2.29) at time T such that :

$$s_t = 0 \qquad \text{for all } t < T.$$

$$s_t = s > 0 \qquad \text{for all } t \geq T.$$

The subsidy is constant for all t $\geq$ T. The households budget constraints is changed to:

*time t* : $\quad p_t f(n_t) - T_t = M_t^d.$

*time t +1* : $\quad (1 - s_{t+1})\, p_{t+1}^e c_{t+1} = M_t^d.$

The equilibrium condition on the money market is still $M = M_t^d$, and combined with the new budget-constraint, we have :

$$p_t f(n_t) - T_t = M \quad \Rightarrow \quad (1-s)p_t f(n_t) = M \quad \Rightarrow$$

$$p_t / p_{t+1} = f(n_{t+1})/f(n_t) \quad \text{for all } t \geq T,$$

However at time T, then

$$p_{T-1}/p_T = (1-s)f(n_T)/f(n_{T-1}) \text{ or}$$

$$p_{T-1}/((1-s)p_T) = f(n_T)/f(n_{T-1})$$

The old generation at time T get a subsidy to the price they have to pay for the good, but they do not have to pay the lump-sum tax, since this is paid by the young generation at time T. The increase in the production subsidy change the offer curve :

(2.31) $$U'\{f(n_{t+1})\}f(n_{t+1})/(1-s) = V'(n_t)f(n_t)/f'(n_t)$$

We can again write $n_t$ as a function of $n_{t+t}$ :

$$n_t = F(n_{t+1}, s)$$

The left-hand side of (2.31) is increased when s is increased. Thus F is shifted upwards when s is increased as shown in figure 2.10.

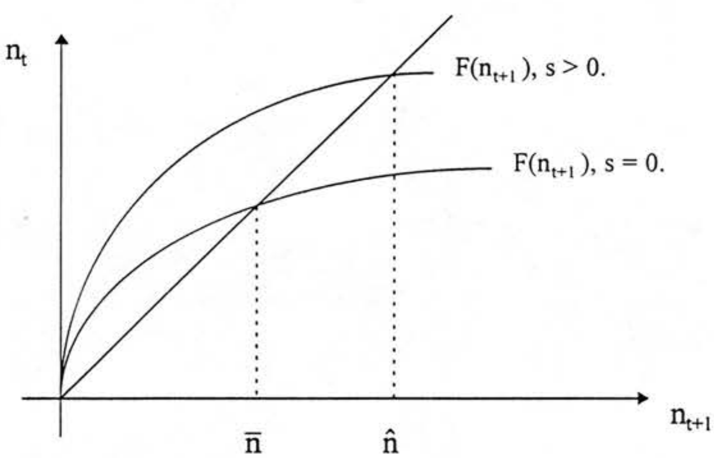


**Figure 2.11.** The upward shift in F as a result of a production subsidy.

The temporary equilibrium condition is given by $n_t = F(n_{t+1}^e, s)$, and the steady state $\hat{n}$ is stable under learning. The welfare comparison between the fast and slow learners are similar to proposition 2.5 where the economy initially is below the steady state, since the initial condition are :

$$n_1^{f,e} = n_1^{s,e} = \bar{n} < \hat{n}.$$

The fast learners are better off initially, but when t is large and the economy is close to the new steady state the slow learners are better off. The steady state welfare decreases although the labour supply increases. The government intervention is still distortionary. However, the households can initially have a higher welfare than the welfare at the steady state $\bar{n}$, this is due to the subsidy to price for the old generation at time T, since the change was unanticipated by the agents at time T. Since the young generation at time T work harder in order to pay the lump-sum tax at time T, the benefits the old generation because $c_T = f(n_T)$, where $n_T > \bar{n}$ and

$$U(c_T) - V(n_{T-1}) = U(f(n_T)) - V(\bar{n}) > U(f(\bar{n})) - V(\bar{n})$$
$$> U(\bar{c}) - V(\bar{n})$$

The old generation at time T have a higher welfare than the steady state level $U(\bar{c}) - V(\bar{n})$. This shown by the peak at t = 50 in figure 2.12. However as t increases the welfare decrease and converge to the new steady state level which is less than $U(\bar{c}) - V(\bar{n})$. The welfare seqeunces are illustrated in figure 2.12.

Figure 2.12 about here.

The figure is generated with a subsidy s = 0.25 and the parametervalues from example 2.1.

## 2.8. Conclusion.

In this chapter, we have studied the welfare consequences for two groups of agents, fast learners and slow learners. We have compared the welfare between the two groups at each time t, the welfare comparison depended on the initial conditions. If the two groups initial were above the stable steady state, the slow learners were initially better off than the fast learners, but for t sufficiently large the fast learners were better off, and remained better off for the rest of the learning transition phase. This result was reversed if the two groups initially were above the steady state.

We introduced different government policies in the model. However, in all three cases the different policies decreased the steady state welfare. If there were multiple equilibria with possible coordination failures, the government can use fiscal policy to move the economy from a low level employment steady state to a high employment steady state. This will be studied in chapter 3 and we will analyse the welfare comparison between the fast learners and slow learners hold, when you move from a low employment steady state to a high employment steady state. We could also introduce a stochastic chock in the production function, this is done in chapter 4, but we would have to make the speed of learning decreasing over time, otherwise the steady state might not be stable under learning. Since the results seem to depend on the learning rule, we should try to analyse cases, where we have different learning rules.

In this model, there is no interaction between the two types of agents, but it would be obvious to change the model such that we have two or more types of agents in the same model. This is done in chapter 5. In chapter 6 we analyse the stability of cycles under learning.

**Appendix to chapter 2.**

*Proof of proposition 2.2(a). The actual interest rate is decreasing.*

$\xi'(n^e_{t+1})$ is given by :

$$\xi'(n^e_{t+1}) =$$

$$\frac{f'(F(n^e_{t+2}))F'(n^e_{t+2})(1-a_t+a_tF'(n^e_{t+1}))f(F(n^e_{t+1})) - f'(F(n^e_{t+1})F'(n^e_{t+1})f(F(n^e_{t+2}))}{f(F(n^e_{t+1}))^2}$$

Hence if

$$f'(F(n^e_{t+2}))F'(n^e_{t+2})(1-a_t+a_tF'(n^e_{t+1}))f(F(n^e_{t+1})) - f'(F(n^e_{t+1})F'(n^e_{t+1})f(F(n^e_{t+2})) < 0$$

then $\xi'(n^e_{t+1}) < 0$. Since $f(n) = An^{\alpha}$ and $F(n) = Bn^{\beta}$ then

$$f'(F(n^e_{t+2}))F'(n^e_{t+2})(1-a_t+a_tF'(n^e_{t+1}))f(F(n^e_{t+1})) - f'(F(n^e_{t+1})F'(n^e_{t+1})f(F(n^e_{t+2}))$$

$$= \alpha A\{(B{n^e_{t+2}}^{\beta})^{\alpha-1}\}\beta B(n^e_{t+2})^{\beta-1}(1 - a_t(\beta B{n^e_{t+1}}^{\beta-1} - 1))A(B{n^e_{t+1}}^{\beta})^{\alpha}$$

$$- \alpha A\{(B{n^e_{t+1}}^{\beta})^{\alpha-1}\}\beta B(n^e_{t+1})^{\beta-1}A(B{n^e_{t+2}}^{\beta})^{\alpha}$$

$$= \alpha A\beta BA(B)^{\alpha-1}(B)^{\alpha}\{(n^e_{t+2})^{\beta\alpha-1}(n^e_{t+1})^{\beta\alpha}(1 - a_t(\beta B{n^e_{t+1}}^{\beta-1} - 1))$$

$$- (n^e_{t+1})^{\beta\alpha-1}(n^e_{t+2})^{\beta\alpha}\}$$

$$= \alpha A\beta BA(B)^{\alpha-1}(B)^{\alpha}(n^e_{t+2})^{\beta\alpha-1}(n^e_{t+1})^{\beta\alpha}\{(n^e_{t+2})^{-1}(1 - a_t(\beta B{n^e_{t+1}}^{\beta-1} - 1))$$

$$- (n^e_{t+1})^{-1}\}$$

$$= \alpha A\beta BA(B)^{\alpha-1}(B)^{\alpha}(n^e_{t+2})^{\beta\alpha-1}(n^e_{t+1})^{\beta\alpha}(n^e_{t+2})^{-1}(n^e_{t+1})^{-1}$$

$$\{\ n^e_{t+1}(1 - a_t(\beta B{n^e_{t+1}}^{\beta-1} - 1)) - n^e_{t+2}\ \}$$

$$= \alpha A\beta BA(B)^{\alpha-1}(B)^{\alpha}(n^e_{t+2})^{\beta\alpha-1}(n^e_{t+1})^{\beta\alpha}(n^e_{t+2})^{-1}(n^e_{t+1})^{-1}$$

$$\{\ n^e_{t+1} - a_t(\beta B{n^e_{t+1}}^{\beta} - n^e_{t+1}) - [\ n^e_{t+1} - a_t(B{n^e_{t+1}}^{\beta} - n^e_{t+1})\ ]\ \}$$

$$= \alpha A\beta BA(B)^{\alpha-1}(B)^{\alpha}(n^e_{t+2})^{\beta\alpha-1}(n^e_{t+1})^{\beta\alpha}(n^e_{t+2})^{-1}(n^e_{t+1})^{-1}\{\ a_tB{n^e_{t+1}}^{\beta}(\beta-1)\ \} < 0.$$

Thus $\xi'(n^e_{t+1}) < 0$.

*Proof of proposition 2.4 (iii).*

The right-hand side of (2.21) is equal to the following expression when we use the C.E.S.-functions from assumption 2.3 :

(2.21)

$$= V'(n_t)\left[\frac{An_t^{\alpha}}{An_{t+1}^{\alpha}}\frac{\alpha An_{t+1}^{\alpha-1}}{\alpha An_t^{\alpha-1}}\beta B(n_{t+2}^e)^{\beta-1}(1 + a(\beta B(n_{t+1}^e)^{\beta-1} - 1)) - \beta B(n_{t+1}^e)^{\beta-1}\right]$$

$$= V'(n_t)\left[\frac{n_t}{n_{t+1}}\beta B(n_{t+2}^e)^{\beta-1}(1 + a(\beta B(n_{t+1}^e)^{\beta-1} - 1)) - \beta B(n_{t+1}^e)^{\beta-1}\right]$$

$$= V'(n_t)\left[\frac{B(n_{t+1}^e)^{\beta}}{B(n_{t+2}^e)^{\beta}}\beta B(n_{t+2}^e)^{\beta-1}(1 + a(\beta B(n_{t+1}^e)^{\beta-1} - 1)) - \beta B(n_{t+1}^e)^{\beta-1}\right]$$

$$= V'(n_t)\beta B(n_{t+1}^e)^{\beta-1}\left[\frac{n_{t+1}^e}{n_{t+2}^e}(1 + a(\beta B(n_{t+1}^e)^{\beta-1} - 1)) - 1\right]$$

$$= V'(n_t)\beta B(n_{t+1}^e)^{\beta-1}\frac{1}{n_{t+2}^e}\left[n_{t+1}^e(1 + a(\beta B(n_{t+1}^e)^{\beta-1} - 1)) - n_{t+2}^e\right]$$

$$= V'(n_t)\beta B(n_{t+1}^e)^{\beta-1}\frac{1}{n_{t+2}^e}\left[n_{t+1}^e + a(\beta B(n_{t+1}^e)^{\beta} - n_{t+1}^e)) - (n_{t+1}^e + a(B(n_{t+1}^e)^{\beta} - n_{t+1}^e))\right]$$

$$= V'(n_t)\beta B(n_{t+1}^e)^{\beta-1}\frac{1}{n_{t+2}^e}aB(n_{t+1}^e)^{\beta}(\beta - 1)$$

Since $\beta < 1$ the last term is negative, and $\partial\omega(n_{t+1}^e, a) / \partial n_{t+1}^e$ is negative.

*Proof of proposition 2.4. The steady state is initally above the steady state.*

*Part 1.* Initially the labour supply of the slow learner and the fast learner is equal : $n_0^s = n_0^f$, since the expected labour supply at time 1 is the same, $n_1^{s,e} = n_1^{f,e}$. At time 1, the actual labour supply of the slow learner is higher than the actual labour supply of the fast learner : $n_1^s > n_1^f$, because $n_2^{s,e} > n_2^{f,e}$. Since $c_1^s = f(n_1^s)$ and $c_1^f = f(n_1^f)$ we have:

$$U(c_1^s) - V(n_0^s) \geq U(c_1^f) - V(n_0^f).$$

*Part 2.* Let us write $U(c_{t+1}) - V(n_t)$ as a function of $n_{t+1}^e$ and a :

$$U(c_{t+1}^s) - V(n_t^s) = \omega(n_{t+1}^{s,e}, a_s)$$

$$U(c_{t+1}^f) - V(n_t^f) = \omega(n_{t+1}^{f,e}, a_f)$$

Since the function $\omega(.,a)$ is a $C^2$-function for all $n^e > 0$ according to assumption 2.1 with a non-vanishing partial derivative of $\omega$ w.r.t. n at $\bar{n}$, $\omega_1(\bar{n},a) \neq 0$, there exists according to the inverse function theorem a unique $C^1$-function $\phi$ such that:

$$x = \phi(\omega(x,a)), \quad \text{for all } x \in N,$$

where N is a neighbourhood around $\bar{n}$.

Let us look at the slow learners first, given the sequence $(n_{t+1}^{s,e})_{t=0}^{\infty}$, determined by

(2.25) $$n_{t+1}^{s,e} = n_t^{s,e} + a_s(F(n_t^{s,e}) - n_t^{s,e}) \quad \text{for all } t \geq 1,$$

and $n_{t+1}^{s,e} > \bar{n}$ for all $t \geq 0$ and $0 < a_s < 1$. Given the expected labour supply sequence, we can construct a welfare-sequence $(\omega(n_{t+1}^{s,e}, a_s))_{t=0}^{\infty} = (\omega_{t+1}^s)_{t=0}^{\infty}$.

We have assumed that the steady state is locally stable under learning, thus $n_{t+1}^{s,e} \to \bar{n}$ for $t \to \infty$, for all $n_{t+1}^{s,e} \in \Omega_1$, where $\Omega_1$ is a neighbourhood around $\bar{n}$. Since $\omega(., a_s)$ is a continuous function, then $\omega_{t+1}^s \to \omega(\bar{n}, a_s) = \bar{\omega}$ when $n_{t+1}^{s,e} \to \bar{n}$. From the inverse function theorem, we can choose a neighbourhood $\Omega_2$ around the steady state $(\bar{n}, \bar{\omega})$ such that :

$$n_t^{s,e} = \phi(\omega(n_t^{s,e}, a_s)) = \phi(\omega_t^s) \quad \text{for all } (n_t^{s,e}, \omega_t^s) \in \Omega_2,$$

This can be inserted into (2.25) and we have a connection between $\omega_{t+1}^s$ and $\omega_t^s$:

(2.26) $$\phi(\omega_{t+1}^s) = \phi(\omega_t^s) + a_s(F(\phi(\omega_t^s)) - \phi(\omega_t^s)).$$

Since $\phi$ is unique, it has an inverse function, and (2.26) can be written as:

(2.27) $$\omega_{t+1}^s = \phi^{-1}[\,\phi(\omega_t^s) + a_s(F(\phi(\omega_t^s)) - \phi(\omega_t^s))\,] = \Phi(\omega_t^s, a_s).$$

It is easy to show that $\overline{\omega}$ is a fixpoint for $\Phi(\,.\,, a_s)$: $\Phi(\overline{\omega}, a_s) = \overline{\omega}$.

Let us now make a first-order Taylor expansion of $\Phi(\,.\,, a_s)$ around $\omega^*$ :

$$\Phi(\omega_t^s, a_s) = \Phi(\overline{\omega}, a_s) + \Phi_\omega'(\overline{\omega}, a_s)(\omega_t^s - \overline{\omega}) + \varepsilon(|\omega_t^s - \overline{\omega}|) \qquad \Rightarrow$$

(2.28) $$\overline{\omega} - \omega_{t+1}^s = \Phi_\omega'(\overline{\omega}, a_s)(\overline{\omega} - \omega_t^s) + \varepsilon(|\omega_t^s - \overline{\omega}|),$$

where $\varepsilon(|\omega_t^s - \overline{\omega}|)/|\omega_t^s - \overline{\omega}|$ converges to 0, when $\omega_t^s$ converges to $\overline{\omega}$, and $\Phi_\omega'(\omega, a_s)$ is given by:

$$\frac{d\Phi(\omega, a_s)}{d\omega} = \frac{d(\phi^{-1}(n))}{dn}[\,\phi'(\omega_t^s) + a_s(F'(\phi(\omega_t^s))\phi'(\omega_t^s) - \phi'(\omega_t^s))\,].$$

When $n = \overline{n}$, we have :

$$\Phi_\omega'(\overline{\omega}, a_s) = (\phi^{-1})'(\overline{n})[\,\phi'(\overline{\omega}) + a_s(\,F'(\phi(\omega_t^s))\phi'(\overline{\omega}) - \phi'(\overline{\omega})\,)\,]$$

$$= \frac{1}{\phi'(\overline{\omega})}\phi'(\overline{\omega})[\,1 + a_s(F'(\phi(\omega_t^s)) - 1)\,] = [\,1 + a_s(F'(\overline{n}) - 1)\,],$$

and (2.28) can be written as follows:

$$\overline{\omega} - \omega_{t+1}^s = [\,1 + a_s(F'(\overline{n}) - 1)\,](\overline{\omega} - \omega_t^s) + \varepsilon(|\omega_t^s - \overline{\omega}|)$$

$$= \theta^s(\overline{\omega} - \omega_t^s) + \varepsilon(|\omega_t^s - \overline{\omega}|)$$

where $1 + a_s(F'(\overline{n}) - 1) = \theta^s$ and $0 < \theta^s < 1$, since $F'(\overline{n}) < 1$.

The above arguments can be repeated for the fast learners, thus given the sequence $(n_{t+1}^{f,e})_{t=0}^{\infty}$, determined by

$$n_{t+1}^{f,e} = n_t^{f,e} + a_f(\,F(n_t^{f,e}) - n_t^{f,e}) \qquad \text{for all } t \geq 1,$$

where $n_t^{f,e} > \overline{n}$ for all $t \geq 1$ and $0 < a_f < 1$. Hence we can construct the welfare-sequence $(\omega(n_{t+1}^{f,e}, a_f))_{t=0}^{\infty} = (\omega_{t+1}^f)_{t=0}^{\infty}$, and $(n_{t+1}^{f,e}, \omega_{t+1}^f) \rightarrow (\overline{n}, \overline{\omega})$ for $t \rightarrow \infty$, because

$\bar{n}$ is locally stable under learning. We can use the inverse function theorem again and there exists a neighbourhood $W_2$ around $(\bar{n}, \bar{\omega})$, where $n_t^{f,e} = \phi(\omega_t^f)$ for all $(n_t^{f,e}, \omega_t^f) \in W_2$. Similar to (2.27) we have the non-linear difference equation:

(2.29) $$\omega_{t+1}^f = \Phi(\omega_t^f, a_f).$$

We can linearise this non-linear difference equation around $\bar{\omega}$ :

$$(\bar{\omega} - \omega_{t+1}^f) = \theta^f(\bar{\omega} - \omega_t^f) + \varepsilon(\bar{\omega} - \omega_t^f)$$

where $\theta^f = (1 + a_f(F'(\bar{n}) - 1)$ so that $0 < \theta^f < 1$, and $\varepsilon(\bar{\omega} - \omega_t^f)/(\bar{\omega} - \omega_t^f) \to 0$ when $\omega_t^f \to \bar{\omega}$. We have $\theta^f < \theta^s$, because $a_s < a_f$ and $F'(\bar{n}) < 1$.

Let $y_t = (\bar{\omega} - \omega_t^f)$ and $x_t = (\bar{\omega} - \omega_t^s)$. Then according to lemma 2.1 there exists an integer $\tau$ such that:

$$(\bar{\omega} - \omega_t^f) < (\bar{\omega} - \omega_t^s) \quad \text{for all } t \geq \tau \quad \Rightarrow$$

(2.30) $$\omega_t^f > \omega_t^s \quad \text{for all } t \geq \tau.$$

From part 1 the slow learner is initially better off at time T, but as part 2 shows the fast learner eventually becomes better off, because he converges faster to the new steady state. ■

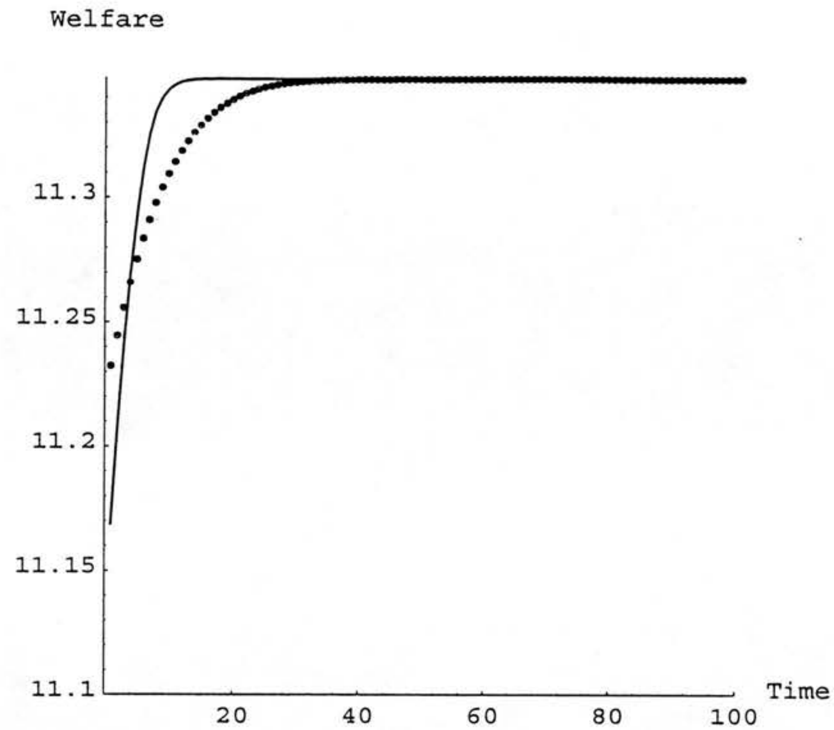


**Figure 2.5 The fast learners welfare (straight line) and the slow learners welfare (dotted line).**

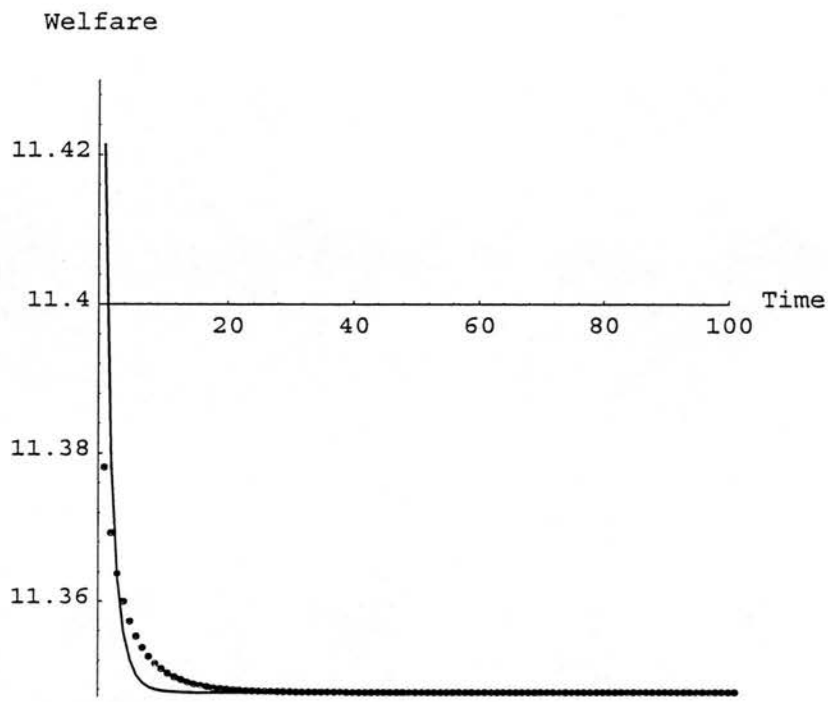


**Figure 2.6 The fast learners welfare (straight line) and the slow learners welfare (dotted line).**

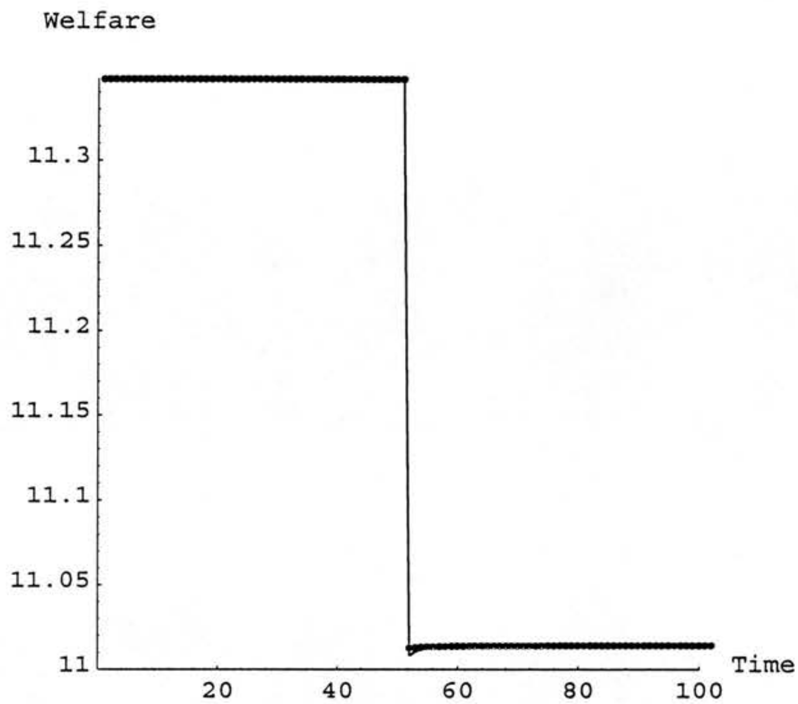


**Figure 2.9. The fast learners welfare (straight line) and the slow learners welfare (dotted line).**

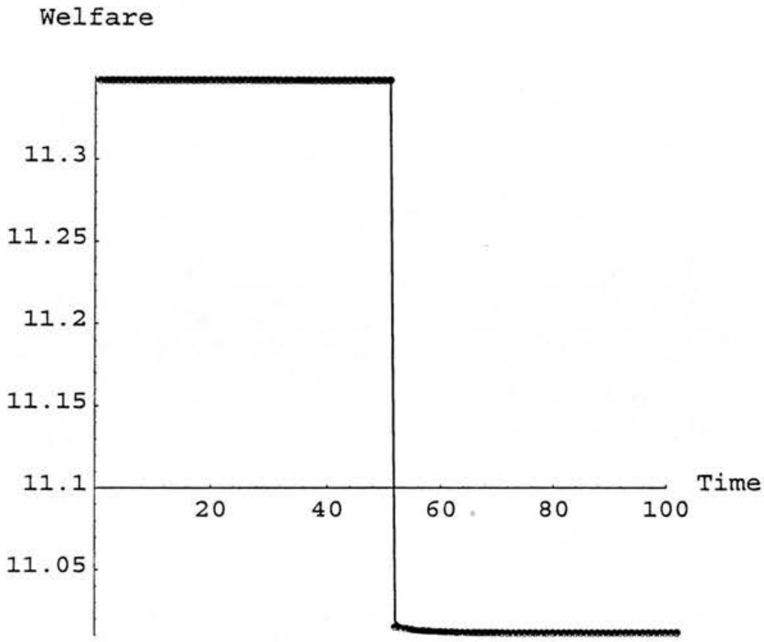


**Figure 2.10. The fast learners welfare (straight line) and the slow learners welfare (dotted line).**

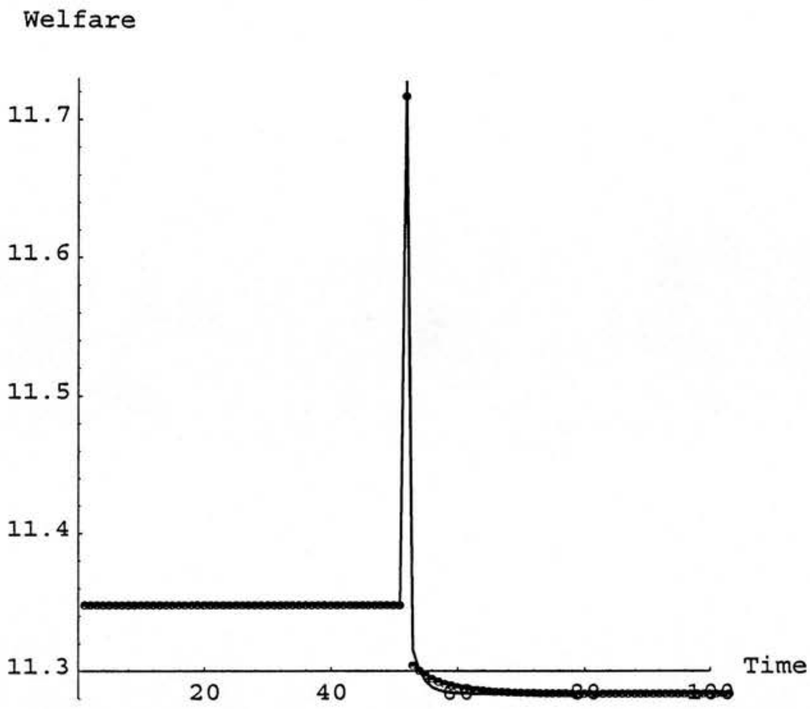


**Figure 2.12. The fast learners welfare (straight line) and the slow learners welfare (dotted line).**

# Chapter 3. Increasing social returns and different speeds of learning.

## 3.1. Introduction.

There has recently been a growing attention into increasing social returns to some factors of production in the analysis of growth and fluctuations in dynamic macroeconomic models, see for example Romer (1986) and Lucas (1988). Increasing returns arise, for example, from learning by doing or search and matching depending on aggregate activity. A different but related approach depends on demand spillovers or imperfect competition, see e.g. Ng (1980), Kiyotaki (1988) and Pagano (1990). A common property of these models are the possibility of multiple equilibria and associated coordination problems, see for example Cooper and John (1988), and Howitt and McAfee (1992).

The possibility of multiple equilibria can be worrying since many of the predictions of the model depend on the choice of a specific member from the set of equilibria. One way of limiting the set of possible rational expectations equilibria is to incorporate learning behaviour in the agents' expectations. In this chapter we analyse the interaction between simple adaptive learning rules and multiple equilibria in a nonlinear model with coordination failures. We will especially study welfare aspects when the government is able to move the economy from a low level equilibrium to a high level equilibrium.

The rationale for government intervention, when there are externalities, would be to move the economy from a low level activity steady state to a high level activity steady state. It is possible to show that the welfare in the different steady states can be ranked such that a low level of activity implies a lower welfare than a high activity steady state. This has been done by Evans and Honkapohja (1995b). Instead of using a policy change to shift between low and high level steady states, we could introduce an exogenous shock in the production function, this is done by Evans and

Honkapohja (1993b). In this chapter we will not introduce the production shock, but focus on different policy changes in a deterministic setting.

We will study the effect of different learning rules, or to be precise, we introduce two types of learners, slow learners and fast learners. The slow learners have a low speed of learning and the fast learners have a high speed of learning. This does not mean that the fast learners learn more about the state of the economy, it simply refers to the weight he use in his forecast and indicate that it is higher than that of the slow learner. Hence the fast learner is making a better forecast than the slow learner. We will compare the labour supply, real interest rate and the welfare between an economy with fast learners and an economy with slow learners, when we move from a low level steady state to a high level steady state. The movement is due to the government introducing a production subsidy financed by a lump-sum tax, this policy will steer the agents towards the high level equilibrium if the subsidy is large enough. It is possible to show that a fast learner is better off in terms of welfare than a slow learner during the movement to the steady state. However, as shown by various simulations this need not be so. If the production subsidy is too high the slow agents overtake the fast agents when the economy is close to the steady state and are better off than the fast agents.

These results are due to three different effects. There is a positive production externality when we increase the production subsidy as a result of the increasing social returns. There is a dynamic learning "effect" from different speeds of learning. However there is also an "inefficient" subsidy effect caused by a very large production subsidy. We will investigate whether the fast learners will be better off compared to slow learners during the entire learning transition, when the agents move towards a high level employment steady state. This assumes that the production subsidy is less than the optimal production subsidy and the speed of learning is not too high. Furthermore we will briefly look at an optimal path for the productions subsidy.

In the literature there has only been few studies into the magnitude of the speed of learning whereby agents learn. Bray and Savin (1986) study the speed of convergence with the help of computer simulations, and with a high speed of learning the agents converges to rational expectations equilibrium. Thus although the agents use a misspecified model when they use the learning scheme they are only misspecified for a short while. In Guesnerie and Woodford (1991) the stability of k-cycles is analysed and it is shown that the stability condition depends on the speed of learning in a rather complicated way. In Vives (1993) there is a study of the factors that influence the speed of learning and the rate of convergence to a rational expectations equilibria.

The chapter is organised as follows. In section 3.2 we describe the model with increasing returns and in section 3.3 we explain the learning rule. The change in government policy is studied in section 3.4 and in section 3.5 we use this analysis to compare the labour supply and real interest rates between a fast learner and slow learner born at the same time. In section 3.6 we look at the welfare comparison between the two groups and investigate if there is an optimal path for the subsidy. Section 3.7 contains the conclusions.

### 3.2. The model.

We analyse a standard overlapping generations model with money, but with increasing social returns in the production function instead of linear production function. The government's policy is a subsidy on the price of output/consumption financed by a lump-sum tax, we could also study government spending financed by a lump-sum tax or by seignorage. The representative agent born at time t maximises welfare $W = U(c_{t+1}) - V(n_t)$, where $c_{t+1}$ is consumption at time t + 1 and $n_t$ is labour supply at time t. The budget constraints for the agent are as follows

$$p_t y_t - T_t = M_t^d \quad \text{and} \quad (1 - s_t)\, p_{t+1}^e c_{t+1} = M_t^d,$$

where $M_t^d$ is the money demand at time t, $p_t$ is the price of output at time t, $p_{t+1}^e$ is the expected price at time t + 1, $y_t$ is the quantity of output produced when the labour supply is $n_t$, $s_t$ is the subsidy rate and $T_t$ is the tax at time t. The government's budget constraint is given by $s_t\, p_t\, y_t = T_t$ such that there is a balanced budget. There is a constant money supply $M_t^s = M$ for all t.

The production function is given by $y_t = f(n_t, N_t)$, where $N_t$ is the aggregate employment given by $N_t = Kn_t$, and K is the number of agents in each generation. The term $N_t$ represents the externality. This type of production function is developed in Evans and Honkapohja (1995b) and is given by

$$f(n_t, N_t) = A(n_t)^{\alpha}\{\max\,(I^*, \frac{\lambda N_t}{1+b\lambda N_t}\}^{\beta} = (n_t)^{\alpha}\psi(N_t),$$

where $A > 0$, $b > 0$, $I^* > 0$, $0 < \alpha < 1$, $\beta > 1$ and $\lambda > 0$. The function f can be illustrated as follows.

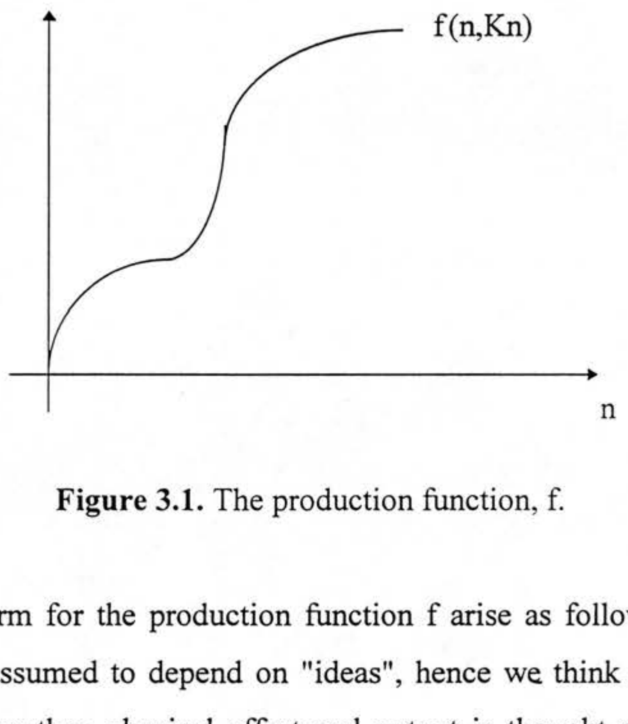


**Figure 3.1.** The production function, f.

This particular form for the production function f arise as follows. The individual agents output is assumed to depend on "ideas", hence we think of labour effort as mental effort rather than physical effort and output is thought of as some type of service based on the ideas. The complementary ideas received from other agents exert a positive externality on productivity, when they are above some threshold, I*.

The number of ideas each agent create is equal the his or her labour supply $n_t$, this also includes additional labour effort. All the ideas are public information and the total number is $N_t = Kn_t$. The ideas are sent at a uniform rate throughout the whole period and agents have a fixed endowment of time to listen to and absorb complimentary ideas. This could be the whole period or some part of the period. Let $\lambda$ be the proportion of "matching" ideas then $(\lambda N_t)^{-1}$ is the amount of time used before a "suitable" idea arrives. However there is a cost of time when the agent absorb a suitable idea, this is assumed to be fixed and equal to the constant b. The total cost of time is equal to $b + (\lambda N_t)^{-1}$ and with the endowment of one unit of time available, the total number of matching ideas the agent have access to is:

$$\frac{1}{b + (\lambda N_t)^{-1}} = \frac{\lambda N_t}{1 + b\lambda N_t}.$$

When the number of complementary ideas received from other agents are above some threshold I* there is a positive externality. The effect generates increasing returns over a range at the aggregate level, however since there is a fixed time cost when absorbing a suitable idea, the range of increasing social returns are bounded. The consumer treats $N_t$ as given in production decisions and with $0 < \alpha < 1$ there is diminishing marginal return to the individual labour supply $n_t$. The production function $f(n_t, N_t)$ is non-decreasing in $N_t$.

The first-order condition for the maximisation problem is given by:

$$(3.1) \qquad U'\left(\frac{p_t f(n_t, N_t) - T_t}{(1 - s_{t+1})p_{t+1}^e}\right) \frac{p_t}{p_{t+1}^e} \frac{1}{(1 - s_{t+1})} f_1'(n_t, N_t) = V'(n_t)$$

The equilibrium conditions are

*Money market.* $\quad M_t^d = M_t^s = M \quad$ for all t.

*Goods market.* $\quad c_t = y_t = f(n_t, N_t) \quad$ for all t.

If the agents had perfect foresight, i.e. $p_{t+1}^e = p_{t+1}$, we have from the budget constraints and equilibrium conditions that $M/p_t = (1 - s_t) f(n_t, N_t)$ in each period, and

$$p_t/p_{t+1} = (1 - s_{t+1}) f(n_{t+1}, N_{t+1}) / (1 - s_t) f(n_t, N_t)$$

If this is inserted in the first-order condition (3.1), we can derive the offer curve describing the evolution of $n_t$ over time :

$$(3.2) \qquad U'(f(n_{t+1}, Kn_{t+1}))f(n_{t+1}, Kn_{t+1}) = \frac{V'(n_t)(1-s_t)f(n_t, Kn_t)}{f_1'(n_t, Kn_t)}.$$

We can find a particular expression for (3.2) given the functions in the following example such that we can simulate the model.

**Example.** *A constant subsidy and C.E.S.-functions.* Let $U(c) = (c^{1-\sigma}/1-\sigma)$, $0 < \sigma < 1$ and $V(n) = (n^{1+\varepsilon}/1+\varepsilon)$, $\varepsilon > 0$ and $s_t = s \in [0,1[$ for all t, then the expression for F(n,s) is as follows :

$$n_t = F(n_{t+1}, s) = [\alpha/(1-s)]^{\frac{1}{1+\varepsilon}}\left[(n_{t+1})^{\alpha}\cdot\psi(Kn_{t+1})\right]^{\frac{1-\sigma}{1+\varepsilon}}$$

where $\psi(Kn_{t+1}) = A(\max\left\{I^*, \frac{\lambda Kn_{t+1}}{1+b\lambda Kn_{t+1}}\right\})^{\beta}$.

This function is S-shaped as long as $0 < \sigma < 1$, see figure 3.2. Since the range of increasing returns are bounded the F-function eventually become concave. If $\sigma > 1$ then F is downward sloping, see Evans and Honkapohja (1995b).

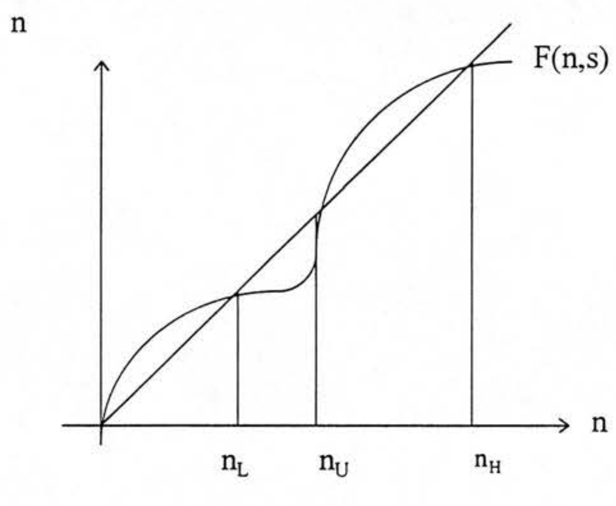


**Figure 3.2.** $n_t = F(n_{t+1}, s)$. [1]

The figure shows three interior steady states and the autarky steady state, where n = 0. Whether there are 3 interior steady states or less depends on the specific parameter values in the example , there could also be one interior steady state or two interior

[1] The parameters values are A = 1, b = 0.2, I* = 2, K = 200, $\alpha = 0.2$, $\beta = 4.8$, $\varepsilon = 0.2$, $\lambda = 0.01$ and $\sigma = 0.41$.

steady states. There are a continuum of perfect foresight equilibrium paths. If F is as in figure 1, then for any $n_0 \in (n_L, n_U) \cup (n_U, n_H)$ there is a equilibrium path converging to $n_U$ and for any $n_0 \in (0, n_L)$ there is a path converging to the autarky steady state.

### 3.3. The learning rule.

Let us now consider the case where the system is not in a perfect foresight equilibrium, and we want to study the evolution of the system when agents are attempting to learn the law of motion, this is done as follows. The agents at time t make a forecast of the price at time t + 1, $p^e_{t+1}$, furthermore the agents observe $p_t$ and $y_t$ in a temporary equilibrium. An agent can thus use $M = p_{t+1}\, y_{t+1}$ or equivalently $p_t / p^e_{t+1} = y^e_{t+1}/y_t$ to forecast on the output $y^e_{t+1}$, or using the production function $y^e_{t+1} = f(n^e_{t+1}, \psi(K n^e_{t+1}))$ the agent can forecast on the labour supply, $n^e_{t+1}$, instead of forecasting on the price in the next period, $p^e_{t+1}$. Here we let the agents forecast the labour supply and the temporary equilibrium condition given by

$$U'(f(n^e_{t+1}, K n^e_{t+1}))f(n^e_{t+1}, K n^e_{t+1}) = \frac{V'(n_t)(1-s_t)\, f(n_t, Kn_t)}{f_1'(n_t, Kn_t)}.$$

This can be rearranged such that

(3.3) $\qquad n_t = F(n^e_{t+1})$.

The agents form expectations of the labour supply according to the learning rule

(3.4) $\qquad n^e_{t+1} = n^e_t + a_t(n_{t-1} - n^e_t) = n^e_t + a_t(F(n^e_t) - n^e_t)$

where $0 < a_t \le 1$ and t = 2,3,.... We assume that $a_t$ satisfy

$$\sum_{t=2}^{\infty} a_t = \infty.$$

This learning rule is also known as econometric learning. The idea behind (3.3) is that the agents devise a formula for the perceived law of motion and use some standard statistical procedure to estimate it. The forecast of the labour supply is calculated from the estimated law of motion. The learning rule is ad hoc but so are most learning rules. However there has been a strong interest in simple learning schemes like this in the literature, see for example Lucas (1985), Marcet and Sargent

(1989) and Woodford (1990), and this learning rule has the merit of simplicity and can be justified by agents' beliefs being "bounded rational" but "reasonable" as mentioned by Guesnerie and Woodford (1991). The agents are boundedly rational in the sense that the model they use is misspecified when they use the learning rule, the reason is that they are outside a rational expectations equilibrium, but in the end the economy converge to a rational steady state.

The learning algorithm can be viewed as an agent acting as if the economy was in a steady state with an unknown value, this value is estimated from past observations using a weighted mean of previous values $n_s$, $s = 2,3,....t-1$ :

$$n_{t+1}^{e} = a_t n_{t-1} + (1-a_t)a_{t-1} n_{t-2} + (1-a_t)(1-a_{t-1})a_{t-2} n_{t-3} + ...$$

In order to make the model simple we assume that $a_t = a$ for all t and $0 < a \leq 1$. This constant a is the *speed of learning* . The speed of learning indicates how much is weight is placed on the difference between the actual labour supply $n_{t-1}$ at time $t-1$ and the expectations $n_t^e$ made at time $t-1$. It would be obvious to use $n_t$ instead of $n_{t-1}$ in the learning algorithm, but we use $n_{t-1}$ to avoid a simultaneous determination of $n_t$ and $n_{t+1}^e$. Furthermore there might be a delay in announcement of labour supply at time t, such that only $n_{t-1}$ is avaible at time t. However it is possible to use $n_t$ at the expense of simplicity.

It might seem strange to learn about the labour supply instead of the price $p_{t+1}^e$ or the real interest rate $(p_t / p_{t+1})^e$ , however in this set-up it is the simplest to do. If the agents learn on the price or the real interest rate it does not alter the results, but there is not a one-to-one correspondence between learning on the labour supply and, for example, learning on the real interest rate. Hence there does not exist a weight $\boldsymbol{a}$ when learning on labour supply that corresponds to a weight $\alpha$ when learning on the real interest rate.

The choice of an adaptive learning rule is not crucial, we could also use a related learning rule formulated in Woodford (1990):

$$n^e_{t+1} = n^e_t + a_t(U'(r_t f(n^e_t, K n^e_t)) r_t f(n^e_t, K n^e_t) - V'(n^e_t))$$

where $r_t = p_{t-1} / p^e_t$. This is sometimes referred to as a gradient rule, see also Sargent (1993). With this algorithm we have the effect on the marginal utility from last periods forecast $n^e_t$, at the steady state $n_H$ : $U'(f(n_H, K n_H)) f(n_H, K n_H) - V'(n_H) = 0$. When $n^e_t > n_H$ the impact on the marginal utility is negative such that agents adjust their expectations downwards and if $n^e_t < n_H$ the impact on the marginal utility is positive such that agents adjust their expectations upwards. This learning rule describes how agents accumulate knowledge about their optimal response to the labour supply.

The stability analysis of the different steady states under learning is developed in Evans and Honkapohja (1995b). It is shown in the case where F are monotonically increasing ($0 < \sigma < 1$):

- If there are three interior steady states as in figure 3.2, then $n_L$ and $n_H$ are stable under learning, while $n_U$ is unstable.
- If there is only one interior steady state ($n_L$ or $n_H$), it is stable under learning.
- If there are two interior steady states ($n_U$ and $n_H$) then $n_H$ is stable under learning and $n_U$ is unstable.

Throughout this chapter, it is assumed that the agents are bounded rational, so the agents treat the economy as if it were in a steady state, the value of which they estimate using the learning rule.

**Example continued.** Given U and V, the function F(.) is

$$n_t = F(n^e_{t+1}) = [\alpha / (1-s)]^{\frac{1}{1+\varepsilon}} [(n^e_{t+1})^{\alpha} \cdot \psi(K n^e_{t+1})]^{\frac{1-\sigma}{1+\varepsilon}}.$$

When $0 < \sigma < 1$ F is S-shaped as in figure 2.

The temporary equilibrium condition (3.3) and the learning algorithm (3.4) provide us with a description of the evolution of the economy over time and the convergence to a steady state depends upon the initial beliefs. Since the two steady states $n_H$ and $n_L$ are the only interior stable steady states, we will focus our analysis on these.

### 3.4. A change in the production subsidy.

Let us consider the situation, where the government changes the production subsidy s

$$s_t = 0 \qquad \text{for } 0 \leq t < T - 1.$$

$$s_t = s > 0 \qquad \text{for } t \geq T.$$

When s = 0, it is assumed that the economy has reached the low level steady state $n^e_{t+1} = n_t = n_L$ for $t = T - 1$ and F is as in figure 3.2. The government's objective is to push the economy to the high employment level $n_H$ by the use of a production subsidy s. This is financed by a lump-sum tax $T_t$ such that the government's budget is balanced: $T_t = sp_t f(n_t)$. The production subsidy s will push the F-curve upwards, because the subsidy makes consumption tomorrow cheaper and since the substitution-effect dominate the income-effect, the household substitute between expensive goods today and cheap goods tomorrow. This can be seen from the F-function in the example that an in increase s, rotates F upwards, see figure 3.3. If the increase in the subsidy s is too small, then $n_L$ still exists and increases together with $n_H$ . The change in s has to be large enough for $n_L$ to disappear and $n_H$ to be the unique interior steady state, see figure 3.3.


**Figure 3.3.** The effect on the F-function of an increase in the subsidy s.

The dynamics are given by the learning rule and the F-function:

$$n^e_{t+1} = n^e_t + a_t(n_{t-1} - n^e_t) \qquad \text{for all t.}$$

$$n_t = F(n^e_{t+1}) \qquad \text{for all t.}$$

What are the effects on the labour supply, real interest rate and private welfare of a change in government policy. It is easy to show that

- The expected labour supply $n^e_{t+1}$ and actual labour supply $n_t$ is increasing and converges to the steady state $n_H$, where s > 0. In the proof for convergence, we use $\sum_{t=2}^{\infty} a_t = \infty$.
- The real interest rate $p_t/p_{t+1} > 1$ for all $t \geq T + 1$, and converges to the steady state level which is equal to 1.

However $p_t/p_{t+1}$ need not be monotone decreasing, because $s_{t+1} = s_t$ and $n_{t+1} > n_t$ for all t > T and the real interest rate is given by :

$$\frac{p_t}{p_{t+1}} = \frac{(1-s_{t+1})f(n_{t+1}, Kn_{t+1})}{(1-s_t)f(n_t, Kn_t)} = \frac{f(n_{t+1}, Kn_{t+1})}{f(n_t, Kn_t)} > 1$$

When $n_t$ and $n_{t+1}$ is in the range where f has increasing returns the difference between $n_t$ and $n_{t+1}$ can be "large" such that the difference between $f(n_t, Kn_t)$ and $f(n_{t+1}, Kn_{t+1})$ is too "large". As t goes towards infinity the difference between $n_t$ and $n_{t+1}$ decreases because $n_t$ converges to $n_H$. When $n_t$ is "close" to $n_H$, both F and f becomes concave and the difference between $f(n_t, Kn_t)$ and $f(n_{t+1}, Kn_{t+1})$ becomes "small", because $n_t$ converges to $n_H$.

Hence the real interest rate might be increasing in the beginning as a result of the increasing returns, but when t is large it has to decrease and converge to the steady state level. If we use the functions from the example we can show this by simulations, see figure 3.4.

Figure 3.4 about here [2]

[2] We use the parameter values from figure 3.2, see footnote 1, with s = 0.25 and $a_t = 0.5$ for all t.

There is an effect from the production externality on the real interest rate, the sequence of the real interest rate, $(p_t/p_{t+1})_t$, is asymptotically decreasing.

### 3.5. The comparison between fast and slow learners.

Let us assume we have two separate economies, one with fast learners and one with slow learners. The speed of learning for the fast learners is given by the sequence $(a_t^f)_t$, where " f " denotes the fast learners, and likewise for the slow learners, $(a_t^s)_t$, where " s " denotes the slow learners. We assume

$$a_s = a_t^s < a_t^f = a_f \qquad \text{for all t.}$$

With this assumption the fast learners make better forecasts than the slow learners as shown in below proposition 3.1, however both types of learners are bounded rational for all t and they are stuck with their learning rule and their respective speed of learning. The assumption that the speed of learning is constant is not crucial, but is made to ease notation. There is no interaction between the two economies as mentioned earlier. Each economy is fully characterised by the labour supply $n_t^i$, the real interest rate $p_t^i / p_{t+1}^i$ and consumption $c_{t+1}^i$ where i = s,f. We compare the expected labour supply between a fast learner and a slow learner born at time t, this is done in the following proposition.

**Proposition 3.1.** *Assume there is a one-time increase in the production subsidy at time T, and let $(n_{t+1}^{f,e})_t$ denote the fast learners' expected labour supply and let $(n_{t+1}^{s,e})_t$ denote the slow learners expected labour supply, then*

$$n_{T+1}^{f,e} = n_{T+1}^{s,e}$$

$$n_{t+1}^{f,e} > n_{t+1}^{s,e} \text{ for all } t \geq T + 1.$$

**Proof.** Since $n_{T+1}^{f,e}$ and $n_{T+1}^{s,e}$ are determined by $n_T^{f,e}$ and $n_T^{s,e}$, respectively, and $n_T^{f,e}$ = $n_T^{s,e} = n_L$ then $n_{T+1}^{f,e} = n_{T+1}^{s,e}$ and $n_T^f = n_T^s$. The rest of the proof is done by induction.

$\mathbf{S^0}$. Since $a_{T+1}^s < a_{T+1}^f$ then

$$n^{f,e}_{T+2} = n^{f,e}_{T+1} + a^{f}_{T+1}(n^{f}_{T} - n^{f,e}_{T+1}) > n^{s,e}_{T+1} + a^{s}_{T+1}(n^{s}_{T} - n^{s,e}_{T+1}) = n^{s,e}_{T+2}.$$

$\mathbf{T^0}$. Assume $n^{f,e}_{t+1} > n^{s,e}_{t+1}$ and show $n^{f,e}_{t+2} > n^{s,e}_{t+2}$ :

$$n^{f,e}_{t+2} = n^{f,e}_{t+1} + a^{f}_{t+1}(n^{f}_{t} - n^{f,e}_{t+1}) = (1 - a^{f}_{t+1})n^{f,e}_{t+1} + a^{f}_{t+1}\, n^{f}_{t}$$

$$> (1 - a^{f}_{t+1})n^{s,e}_{t+1} + a^{f}_{t+1}\, n^{s}_{t} = n^{s,e}_{t+1} + a^{f}_{t+1}(n^{s}_{t} - n^{s,e}_{t+1})$$

$$> n^{s,e}_{t+1} + a^{s}_{t+1}(n^{s}_{t} - n^{s,e}_{t+1}) = n^{s,e}_{t+2} ,$$

where the first inequality is due to $n^{f,e}_{t+1} > n^{s,e}_{t+1}$ and the second $a^{s}_{t} < a^{f}_{t}$. ■

The fast learners have according to proposition 3.1, a better forecast than the slow learners when we change the production subsidy. The fast learners respond more rapidly to a change in the subsidy. It is easy to see from proposition 3.1 that the actual labour supply for a fast learner born at time t > T is greater than the actual labour supply of a slow learner :

$$n^{f}_{t} > n^{s}_{t} \qquad \text{for all } t > T.$$

At time t = T, $n^{f}_{T} = n^{s}_{T} > n_L$ because $n^{f,e}_{T+1} = n^{s,e}_{T+1} = n_L$. Hence there is an immediate increase in the actual labour supply at time T, but the effect from different speeds in the learning rule does not have an impact on the expected labour supply before time T + 1. The agents does not update their beliefs before time T + 1. The increase in the expected labour supply is due to the monotonicity of the F-function. The agents keep working harder each period, and converge to the new steady state with high activity. Let $r^{f}_{t+1} = p^{f}_{t} / p^{f}_{t+1}$ denote the fast learner's real interest rate and let $r^{s}_{t+1} = p^{s}_{t} / p^{s}_{t+1}$ denote the slow learner's real interest rate, then we have the following proposition, where we compare the real interest rates.

**Proposition 3.2.** *Assume there is a one-time increase in the subsidy, then*

$$r^{f}_{T+1} > r^{s}_{T+1} ,$$

*and there exists an integer $\tau$ such that*

$$r^{f}_{t+1} < r^{s}_{t+1} \text{ for all } t \geq \tau.$$

**Proof.** See appendix. ■

Initially the fast learners have a higher real interest rate, but when t is large the slow overtakes the fast learners such that they have a higher real interest rate. This is due to the difference in the speed of learning and the fact that the two economies do not interact, if you are a slow learner then the generation before you were slow and the generations to come will be slow.

It is possible to compare the welfare at the different steady state levels for a given subsidy :

- If $n_H > n_L$ then the welfare at $n_H$, $U(n_H) - V(n_H)$, is higher than the welfare at $n_L$, $U(n_L) - V(n_L)$, for given subsidy $s \geq 0$.

For a proof of this, see Evans and Honkapohja (1995b). When the subsidy s is changed the steady states are changed and we cannot make the same comparison between a steady state where $s = s_1$ with a steady state where $s = s_2$ and $s_1 \neq s_2$.

However the low activity level $n_L$ , where $s = 0$, is "locally" pareto optimal. If we have a small increase in s which does not remove $n_L$, but simply increases to $\tilde{n}_L > n_L$ , then the welfare $U(\tilde{c}_L) - V(\tilde{n}_L)$ is less than $U(c_L) - V(n_L)$. This is due to the inactivity of the production externality at this level. If the government introduces a small subsidy s when the economy are at $n_L$ , then although $n_L$ increase, the welfare will decrease in the new steady state, the reason is that we have not reached the range of increasing returns. The change in the production subsidy has a purely distortionary effect.

If there is a fixed subsidy s, the welfare is increased when n is increased due to the ranking of the steady states. From this positive externality, the welfare at $n_H$ when $s = 0$, is less than the welfare at $n_H$ where $s > 0$, because an increase in the subsidy will activate the production externality. Hence a small increase in the subsidy when the economy initially are at $n_H$ will raise the welfare in the new steady state. However the government cannot raise s arbitrarily high. If the increase in s is too high then the level of welfare at $n_H$ with s "high" can even be lower than the welfare at $n_H$ with s = 0. There is an optimal production subsidy s*, where welfare is maximal, and s* is determined as a solution to the following maximisation problem

$$\max_{n} \; U(f(n, Kn)) - V(n)$$

The first-order condition for this problem is:

$$U'(f(n,Kn))( f_1(n,Kn) + f_2(n,Kn)\partial\{\psi(Kn)\}/\partial n ) - V'(n) = 0 \tag{3.5}$$

where $f_1$ is the partial derivative f with respect to the first argument and $f_2$ is partial derivative of f w.r.t. the second argument. If we insert the expression of the offer curve (3.1), we have

$$U'(f(n,Kn))( f_1(n,Kn) + f_2(n,Kn)\partial\{\psi(Kn)\}/\partial n )$$
$$- U'(f(n,Kn))f_1(n,Kn)/(1 - s^*) = 0 \Rightarrow$$

$$s^* = f_2(n,Kn)\frac{\partial\psi(Kn)}{\partial n}\{ f_1(n,Kn) + f_2(n,Kn)\frac{\partial\psi(Kn)}{\partial n} \}^{-1} \tag{3.6}$$

where $n = n_H$.

Let us assume that F is as in figure 3.2 with s = s*. If the economy is at $n_L$ and the government wishes to bring the economy to $n_H$, it has to increase s such that s > s* and shortly later it must decrease the subsidy s to s = s*. The increase in s reduces the steady state welfare compared with the steady state welfare at $n_H$ where s = s*, and if the increase in s is too high the welfare might even be lower than at $n_L$. There are several problems with this increase, for example, for how long is it necessary for the government to have s > s* and what is the optimal path for the subsidies when the agents use an adaptive learning rule. The government wants to minimise the time where s > s*, but the agents need to get above $n_H$ where s = s*, before the government decreases s again. If the government reduces s to s* before the level of labour supply is above $n_H$ (s = s*), then they converge back to the old level $n_L$. This will among other things depend on the speed in the learning rule, if agents have a low speed learning rule it might take them some time before they are above $n_H$.

### 3.6. The welfare comparison between the two economies.

In this section we will compare the welfare between an economy with slow learners and an economy with fast learners. Let $(n_{t+1}^{f,e})_t$ denote the fast learners labour supply and let $(n_{t+1}^{s,e})_t$ denote the slow learners labour supply. Given these two sequences we can calculate the labour supply, the price and the consumption for both economies, remember there is no interaction between the economy with fast agents and the economy with slow agents. We wish to compare the welfare after a change in the production subsidy at time T, between a slow learner born at time t, $U(c_{t+1}^{s}) - V(n_{t}^{s})$, and a fast learner born at time t, $U(c_{t+1}^{f}) - V(n_{t}^{f})$ for all $t \geq T$.

At time T, the old generation of fast learners has the same welfare as the old generation of slow learners, however there is an increase in the welfare compared to the old generation at time T – 1:

$$U(c_T^f) - V(n_{T-1}^f) = U(c_T^s) - V(n_{T-1}^s) > U(c_{T-1}) - V(n_{T-2}) = U(c_L) - V(n_L).$$

The increase in welfare is due to the young generation at time T is working harder, $n_T^i > n_{T-1}^i$ for i = s,f. This is a result of the production subsidy having an immediate effect on the F-function, see figure 3.3. However the forecast at time T $n_{T+1}^{i,e}$, i = s,f, is not affected because it depends on the forecast at time T –1, $n_T^{i,e}$, and the labour supply at time T – 1, $n_{T-1}^i$, and the change was unanticipated by the agents. Hence there is a one-period delay in revising the forecast after the change in the production subsidy. The old generation at time T benefits from the increase in the actual labour supply $n_T$ because the consumption $c_T = f(n_T, Kn_T)$.

Furthermore there is a decrease in the price the old generation pays for the consumption good at time T due to the production subsidy s :

$$p_{T-1}/(1-s)p_T = f(n_T, Kn_T)/f(n_{T-1}, Kn_{T-1}) > 1 \qquad \Rightarrow$$

$$p_{T-1} > (1-s)p_T \ .$$

At time t = T + 1, the welfare for a fast learner is higher than the welfare for a slow learner :

$$U(c^f_{T+1}) - V(n^f_T) > U(c^s_{T+1}) - V(n^s_T),$$

since $n^f_T = n^s_T$ and $c^f_{T+1} > c^s_{T+1}$. Here the effect of higher speed of learning benefits the fast learner, because the effect of different speeds of learning make the young generation of fast learners at time T + 1 work harder than the young generations of slow learners at time T + 1 : $n^f_{T+1} > n^s_{T+1}$, while at time T , $n^f_T = n^s_T$.

When we compare the two economies at time T + 1 it is helpful to use simulations. In the simulations we use the functions from the example and parameter values in footnote 1. The simulations indicate that there are two different cases :

- The fast learners are better off than the slow learners for all $t \geq T$, see figure 3.5 and figure 3.6.
- The fast learners are better off than the slow learners initially while the slow learners are better off when the economy is close to the steady state, see figure 3.7.

Figure 3.5 about here.

We use the parameter values from figure 3.2 and set s = 0.25 and $a_f = 0.5$, $a_s = 0.2$. Here we have chosen a subsidy large enough to remove the low level steady state. The difference in the speed of learning could be made smaller without changing the results, but this obviously depends on the parameter values we have chosen in the example. The two welfare-sequences, given by $\{U(c^s_{t+1}) - V(n^s_t)\}_t$ and $\{U(c^f_{t+1}) - V(n^f_t)\}_t$ are monotone and converge to the steady state level $U(c_H) - V(n_H)$ and the fast agents are always better of than the slow agents for all t.

Figure 3.5 shows the positive effect on the welfare from the increasing social returns, and both types of agents benefits from this effect. If the production function were strictly concave, the steady state welfare would decrease, although the steady state value of n increases, this is due to the distortionary effect from the increase in s.

Here we have an increase in the welfare when the economy is pushed from $n_L$ to $n_H$, given a change in the subsidy s is large enough to eliminate $n_L$.

- There is a static positive externality effect from an increase in the subsidy $s < s^*$, when the production function exhibits increasing social returns.

There is also a dynamic effect from the difference in the learning rules, as can be seen from figure 3.5, the fast learner born at time t have a higher welfare compared to a slow learner born at time t :

$$U(c^f_{t+1}) - V(n^f_t) > U(c^s_{t+1}) - V(n^s_t) \quad \text{for all t.}$$

This is a result of the increase in the level of steady state welfare together with the fast learner making a better forecast than the slow learner. The welfare is increasing due to the increasing social returns, and the welfare sequences for the two economies are monotonically increasing. Since the fast learners are "closer" to the steady state :

$$| n^{f,e}_{t+1} - n_H | < | n^{s,e}_{t+1} - n_H | \qquad \text{for all } t > T$$

they have higher welfare compared to the slow agents who do not adjust their forecast as fast.

If the production function is strictly concave, the fast learner is better off initially, but when t is large the slow learner overtakes the fast learner. This is a result of the difference in the speed of learning : The fast learner changes his forecast more rapidly than the slow learner, but because of the decrease in the steady state level of welfare, the fast learner converge to the steady state " before " the slow learner. Hence even though the fast agent make a better forecast, he is worse off when t is large, given a strictly concave production function, because the steady state welfare was decreasing and the fast was closer to the steady state. This is also possible if there is increasing social returns in the production function as will be shown below in figure 3.7.

In the following experiment, shown in figure 3.6 below, we have increased the subsidy to s =0.6 but left the remaining parameters unchanged from figure 3.5.

Figure 3.6 about here.

As can be seen from figure 3.6, the fast learner is still better off than the slow learner for all t > T + 1, but the welfare sequence for the group of fast learners, $\{U(c_{t+1}^{f}) - V(n_{t}^{f})\}_{t}$, is no longer monotone as in figure 3.6. This is due to the increase in s compared to figure 3.5, such that the range over which we have increasing returns has increased. This can lead to a large difference between $n_{t}^{f}$ and $n_{t+1}^{f}$, and since the consumption $c_{t+1}^{f} = f(n_{t+1}^{f}, Kn_{t+1}^{f})$, it is possible that the welfare $U(c_{t+1}^{f}) - V(n_{t}^{f})$ can be above the steady state level $U(c_H) - V(n_H)$. However as t goes to infinity the welfare converges to the steady state level, because the steady state is stable under learning rule. Hence at some time $\tau$ the welfare of the fast learners must begin to decrease and converge to the steady state level. the fast learners are better compared to the slow learners during the learning transition. Furthermore there is also an effect from the speed of learning, since the slow learners still have a monotone increasing welfare sequence, $\{U(c_{t+1}^{s}) - V(n_{t}^{s})\}_{t}$, but an increase in $a_s$ will make the welfare sequence non-monotonic, if the increase is large enough. Hence the combination of a "large" subsidy s and a "high" speed of learning can cause the welfare sequence to be non-monotonic as shown in figure 3.6. In figure 3.6 and in figure 3.5 there is a "welfare effect" for the old generation at t = 50, they get the subsidy to price, but do not have to pay the tax at time t = 50.

It is possible that the welfare for both types of learners is decreasing in the first periods after the change in the subsidy. This is due to the "local" pareto-optimality of $n_L$, when the labour supply has not reached the range of increasing returns. This is also possible even if the subsidy is large enough to eliminate $n_L$, as can be shown by simulations. The "slower" the agents are, the longer it takes them to move to the range where the production function exhibits increasing returns.

In figure 3.7 we have increased the subsidy to s = 0.8 and this above the optimal subsidy $s^* \approx 0.62$, but the remaining parameters are as in figure 3.5 and 3.6.

Figure 3.7 about here.

Here the subsidy is higher than the optimal subsidy, the steady state level of welfare at $n_H$, where s = 0.8 is lower compared to the steady state welfare at $n_H$, where s = 0.6 see figure 3.6. Hence there is an "inefficiency" effect from a very large increase in the subsidy. As in figure 3.6, the welfare sequence is non-monotonic but now it is for both groups of learners, and even if $a_s$ is "very small" the welfare sequence remains non-monotonic. There is again a "welfare effect" at t = 50, the old generation at t = 50 benefits from the subsidy and do not pay the tax.

As figure 3.7 shows, when t is sufficiently large the slow learner has a higher welfare than the fast learner, even though the fast learner is making a better forecast than the slow learner as shown in proposition 3.1. This might seem counterintuitive that somebody with a better forecast could do worse than somebody with a "worse" forecast, but remember there is no interaction between the two economies. Furthermore if the generation before was slow (or fast) learners, the generations to come will also be slow (or fast) and it takes the slow learners longer time to converge to the steady state compared to the fast learners as shown in proposition 1. Since the welfare for both groups is above the steady state level from some time $\tau$, the slow learners will eventually overtake the fast learners when t is large, while initially the fast learners are better off than the slow learners. Hence there are two externalities working together here:

- There is a static inefficiency effect from $s > s^*$, even though the change in s increases the labour supply, the welfare in the steady state with $s > s^*$ decreases compared to the welfare in the steady state where $s = s^*$.
- There is a dynamic learning effect from $a_f > a_s$, and although $n_t$ increases, the welfare sequence is not monotonically increasing. When t is above some integer $\tau$, the welfare begins to decrease and converges to the steady state level. Since the slow learners do not converge as rapid as the fast learners, they eventually overtake the fast learners when t is sufficiently large.

We can now show the following local result.

**Proposition 3.3.** *Assume the subsidy is increased from $s = 0$ to $s > 0$.*

*(i) If the welfare of the fast and slow learners, $U(c^i_{t+1}) - V(n^i_t)$, for $i = s,f$, is below the steady state level of welfare $U(c_H) - V(n_H)$ for all t, then there exists an integer $\tau$ such that*

$$U(c^s_{t+1}) - V(n^s_t) < U(c^f_{t+1}) - V(n^f_t) \qquad \text{for all } t > \tau.$$

*(ii) If there exists an integer m such that the welfare of the fast and slow learners, $U(c^i_{t+1}) - V(n^i_t)$, for $i = s,f$, is above the steady state level of welfare $U(c_H) - V(n_H)$ for all $t > m$, then there exists an integer $\tau > m$ such that*

$$U(c^s_{t+1}) - V(n^s_t) > U(c^f_{t+1}) - V(n^f_t) \qquad \text{for all } t > \tau.$$

**Proof.** See appendix. ■

In order to determine if we are in case (i) or in case (ii), we have to find the condition for $U(c^f_{t+1}) - V(n^f_t) < U(c_H) - V(n_H)$ for all t, and this depends upon the size of the subsidy s and the speed of learning a. If we are in case (ii), we have to find the criterion for $U(c^s_{t+1}) - V(n^s_t) > U(c_H) - V(n_H)$ for t sufficiently large, this will also depend on the size of s as well as the speed of learning.

The second statement in proposition 3.3 illustrates the possibility of slow learners being better off than fast learners. The simulations and the above analysis suggests a more general proposition. After a one-time increase in the subsidy s, but where the subsidy is less than the optimal subsidy, $s < s^*$, if furthermore the slow learners speed of learning is less than some constant, $a_s < k_1$, then the welfare of the slow agents is below the steady state welfare:

$$U(c^s_{t+1}) - V(n^s_t) < U(c_H) - V(n_H) \text{ for all t}$$

The combination of a "low" subsidy and a "low" speed of learning makes the welfare sequence monotonically increasing and bounded above by the steady state welfare during the learning transition. The fast agents' welfare sequence might also be monotonically increasing and bounded above, then we should be able to prove by induction that the fast learners are better off than the slow learners, since the fast agent is "closer" to the steady state as in figure 3.5. In the case where the agents welfare sequence increases above the steady state at some point $\tau$ because $a_f > k_1$ then

$$U(c_H) - V(n_H) < U(c^f_{t+1}) - V(n^f_t) \quad \text{for all } t > \tau,$$

and the fast learner is still better off compared to the slow learners, see figure 3.6. In both situations we can use the first part of proposition3.3.

In the last case where the slow agents are better off than the fast agents when t is large as in figure 3.7, then a high subsidy, $s > s^*$, or a high speed learning for the slow learners, $a_s > k_2$, makes the welfare-sequences non-monotone at some point and the welfare increase above the steady state level :

$$U(c_H) - V(n_H) < U(c^i_{t+1}) - V(n^i_t) \quad \text{for } i = s,f.$$

In this case we can use the second part of proposition 3.3. Hence the following conjecture should cover all three situations.

**Conjecture.** *Assume there is an increase in s from $s = 0$ to $s > 0$.*

*(i) If $s < s^*$ and $a_s < k_1$, where $k_1$ is some constant, then the fast learners are always better off than the slow learners :*

$$U(c^s_{t+1}) - V(n^s_t) < U(c^f_{t+1}) - V(n^f_t) \qquad \text{for all } t.$$

*(ii) If $s > s^*$ or $a_s > k_2$, where $k_2$ is some constant, then the slow learners overtake the fast learners and there exists an integer $\tau$ such that :*

$$U(c^s_{t+1}) - V(n^s_t) > U(c^f_{t+1}) - V(n^f_t) \qquad \text{for all } t > \tau.$$

The problem with a conjecture as the above, is that if we assume that U and V are C.E.S.-functions as in the example, the constants $k_1$ and $k_2$ may depend on the speed of learning and subsidy as well as the other parameters, furthermore the welfare function is not concave over a range of values for $n^e_{t+1}$. Hence it is not as straightforward to show the conjecture. as one might expect, even in the case were we assume that U and V are C.E.S. functions.

With this in mind let us study the policy problem where the government wants to increase the subsidy in order to steer the economy away from the low level activity steady state to the high level steady state. Instead of comparing the welfare of a fast learner with the welfare of a slow learner at each point in time, we will compare the total welfare, i.e. the sum of all the agents' individual welfare during the learning transition, for different government policies, in order to find the optimal path for the subsidy $s_t$ over time.

We assume that the economy is at the point A in figure 8 below, at the low level of activity steady state with s = 0. The government would like to move the economy to B, where s = s*, the optimal subsidy, and the steady state welfare is maximal. In order to push the economy from A to B the subsidy is raised to $s_1$ , this rotates the F-function upwards as we have seen previously, since the substitution-effect dominates the income-effect. The government will thereafter reduce the subsidy to $s_2 = s^*$ after some time as shown in figure 3.8. This pushes the F-function downwards with the steady state at B.

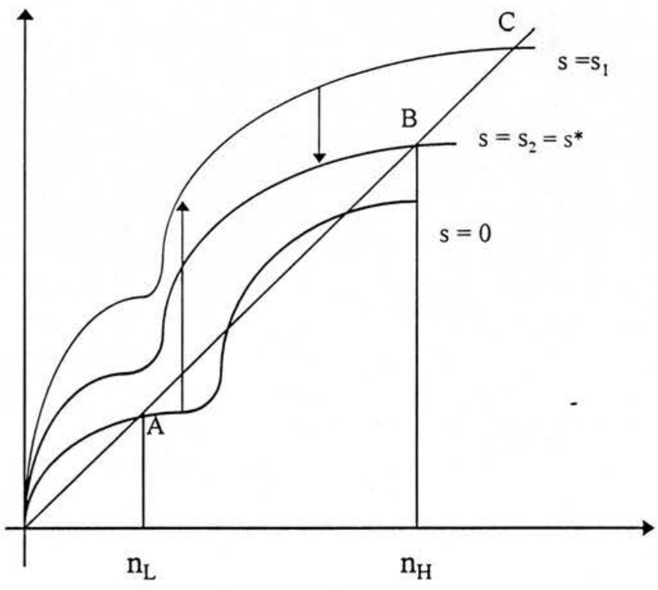


**Figure 3.8.** The effect on F with an initial increase in the subsidy from s to $s_1$ and a subsequent decrease to $s_2 = s^*$.

Since there is no discounting in the model, we can evaluate the policy by the finite sum of each generations welfare:

$$(3.7) \qquad \sum_{t=T}^{N} U(c_{t+1}^{i}) - V(n_t^i) \qquad \text{for } i = s,f.$$

Where N is a large but finite integer and T is the time of the first change in the subsidy. If we let N go to infinity the sum will become infinite, because it is a sequence of positive numbers that converges to $U(c_H) - V(n_H) > 0$. One way to overcome this problem would be to discount future generations welfare by a discount-factor, for example, the real interest rate $r_{t+1}$. Another possibility is to look at the infinite sum but take the difference the welfare during the learning transition ad the steady state:

$$(3.8) \qquad \sum_{t=T}^{\infty} \left\{ (U(c_{t+1}^{i}) - V(n_t^i)) - (U(c_H) - V(n_H)) \right\}$$

Since $n_t^i$ converge to the steady state level it should be possible to prove that (3.8) was well-defined, but it depends on whether it is possible to find a function $g(n_t)$ such that

$$\sum_{t=T}^{\infty} \left\{ (U(c_{t+1}^{i}) - V(n_t^i)) - (U(c_H) - V(n_H)) \right\} < \sum_{t=T}^{\infty} g(n_t) < \infty.$$

However in order to keep the set-up simple let us for the moment merely look at the finite sum. Since there are lots of policies that end up at B where $s = s^*$, we have to find the optimal path for the subsidy, this could, for example, be :

- **Policy a.** An increase in s to $s^*$ at time T : $s_t = s^*$ for all $t \geq T$.
- **Policy b.** An increase in s to $s_1 > s^*$ for $t \geq T$ until some time $\tau$, where s is reduced to $s_2 = s^*$ :

$$s_t = s_1 \qquad \text{for } T \leq t \leq \tau - 1$$

$$s_t = s_2 = s^* \qquad \text{for } t \geq \tau.$$

We do not want too many changes in the subsidy, because then the agents would take these changes into account when they make their forecast. Until now the agents take $s_t$ as given in every period and they expect next periods subsidy to be equal to the previous period's subsidy and we keep this assumption for simplicity. We have only studied unanticipated changes in the production subsidy, but it would be a natural extension to incorporate expectations about the governments policy as well.

If we look at **policy a,** the welfare increases due to the positive production externality. If the agents speed of learning is to high then it may rise above the steady state level but has to decrease since the learning rule takes them to the steady state $n_H$.

With the **policy b**, the increase in s to $s_1$ leads to a higher steady state level of employment but a lower steady state welfare compared to the steady state labour supply and welfare when s = s*, this is due to s being increased above the optimal level s*. The government has to decide for how long it wishes to have $s_t = s_1$ . It is necessary to have $s_1$ for at least as many periods as it takes $n_t^e$ to move above a possible low activity steady state $n_L$ where s = s*. Otherwise the economy will simply converge to this new low level activity steady state, where the welfare is lower than welfare in the old low level activity steady state, because of the local pareto-optimality of $n_L$ with s = 0. A way to ensure that the government would not run into this type of problem would be to have $s_t = s_1$ until $n_t^e$ is above the steady state $n_H$ where s = s*, i.e. that we would have a large $\tau$ in **policy b**. However a large $\tau$ could reduce the welfare during the learning transition, but a large subsidy combined with high speed of learning will increase the labour supply "very fast" such that it is above the steady state $n_H$ where s = s* within few periods after the initial change. Hence we have to consider the possible loss of welfare due to a large $\tau$, against a quick response due to a high speed of learning and a large subsidy. When s is decreased to s*, this will lead to a decrease in $n_t$ but an increase in the welfare, because the subsidy is at its optimal level.[3]

The question is now, which of the two types of policies gives the highest welfare during the learning transition, and does the optimal path of subsidies depend on a, the

[3] The analysis where s is decreased is similar to the previous, where s is increased.

speed of learning. The policy we are looking for must give a higher welfare during the transition from A to C and from C to B rather than staying at A, but how big should $s_1$ be and how fast can we reduce it to $s_2$ . It is easy to show the following result :

- A policy with a one-time change in s from s = 0 to s > 0, but s ≤ s*, where the increase is large enough to eliminate $n_L$ will always give a higher welfare than staying at A : $\sum_{t=T}^{N} U(c^i_{t+1}) - V(n^i_t) > \sum_{t=T}^{N} U(c_L) - V(n_L)$
  for N sufficiently large and i = s,f.

When the government increase the subsidy, $n_t$ will eventually reach the range where there is increasing returns and where it converges to a high level steady state with higher welfare than the welfare in the old low level steady state $n_L$ . Thus the government just need to choose N sufficiently large. Hence if s is increased to s* at time T, this policy will give a higher welfare for the group as a whole rather than staying at A, where s = 0 and n = $n_L$ .The government would thus be induced to introduce the subsidy in this case, however it might not be a trivial task to calculate s* as can be seen from the expression in (3.6), because f is not concave thus the maximisation problem is non-concave. The problem become even worse if the economy is at $n_L$ with s = s*, then an increase in s reduces the steady state level of welfare at $n_H$ as discussed earlier. We need to evaluate :

**(a)** The policy where $s_t$ = s* for t ≥ T :

$$\sum_{t=T}^{N} U(c^i_{t+1}) - V(n^i_t)\,.$$

**(b)** The policy where $s_t = s_1$ for $T \leq t \leq N_1$ and $s_t = s_2$ for $t \geq N_1 + 1$ :

$$\sum_{t=T}^{N_1} U(c^i_{t+1}) - V(n^i_t) + \sum_{t=N_1+1}^{N} U(c^i_{t+1}) - V(n^i_t)\,.$$

Which of these policies give the highest welfare will depend on the size of the subsidy and the size of the speed of learning and again it might not as straightforward to show which policy give the highest welfare, since functions are not concave for all values of the expected labour supply.

### 3.7. Conclusion.

In this chapter we have studied the effects on increasing social returns in a standard model with overlapping generations, where the agents forecast on the labour supply. This lead to multiple equilibria with different levels of activity. The learning behaviour could be used in the model narrow down set of "reasonable" equilibria where the outcome depended on the initial conditions. The purpose was to study the effect of changes in government policy on two different groups of learners, where each group was determined by their speed of learning. The group with the higher speed of learning made better forecast than the group with a low speed of learning in the sense that the fast learners converged more rapidly to the new equilibrium compared to the slow learners.

In most cases this leads to a higher welfare for the fast learner compared to the slow learner because of the positive production externality and higher speed of learning. However it was possible that the slow learners were better off when the economy is close to the steady state, see proposition 3.3. This was due to the dynamic learning externality together with a "large" subsidy s such that the welfare was increased above its steady state level. Hence it might not always be favourable to make "better" forecasts.

**Appendix to chapter 3.**

*Proof of proposition 3.2.* The real interest rate $r_{t+1} = p_t / p_{t+1}$ can be written as a function of the expected labour supply $n^e_{t+1}$ :

$$r_{t+1} = \frac{p_t}{p_{t+1}} = \frac{f(n_{t+1}, Kn_{t+1})}{f(n_t, Kn_t)} = \frac{f(F(n^e_{t+2}), KF(n^e_{t+2}))}{f(F(n^e_{t+1}), KF(n^e_{t+1}))} = \phi(n^e_{t+1}),$$

because $n^e_{t+2} = n^e_{t+1} + a(F(n^e_{t+1}) - n^e_{t+1})$. $\phi(.)$ is a $C^2$-function. Hence for the fast and slow learner, we have :

$$r^f_{t+1} = \phi(n^{f,e}_{t+1}) \quad \text{and} \quad r^s_{t+1} = \phi(n^{s,e}_{t+1}).$$

Since $n^{f,e}_{t+1} \to n_H$ when $t \to \infty$ and $n^{s,e}_{t+1} \to n_H$ when $t \to \infty$, then the real interest rates converge as well : $r^f_{t+1} \to 1$ and $r^s_{t+1} \to 1$ when $t \to \infty$, remember that F and f eventually become concave and remains concave for all n above a certain value, especially both F and f become concave for n in a neighbourhood around $n_H$. Let $r_H = 1$ denote the steady state real interest rate. Since $\phi$ is a $C^2$-function with $\phi'(n_H) \neq 0$, there exists a neighbourhood $\Omega$ around $(n_H, r_H)$ such that

$$n^{e,f}_{t+1} = \psi(r^f_{t+1}) \text{ for all } (n^{f,e}_{t+1}, r^f_{t+1}) \in \Omega$$

according to the inverse function theorem. If this is inserted in the learning rule we have

$$\psi(r^f_{t+2}) = \psi(r^f_{t+1}) + a_f(F(\psi(r^f_{t+1})) - \psi(r^f_{t+1}))$$

Since $\psi$ is unique, it has an inverse function and

$$r^f_{t+2} = \psi^{-1}\{\psi(r^f_{t+1}) + a_f(F(\psi(r^f_{t+1})) - \psi(r^f_{t+1}))\} = \Psi(r^f_{t+1}).$$

It is easy to show that $r_H$ is a fixpoint for $\Psi(.)$ :

$$\Psi(r_H) = \psi^{-1}\{\psi(r_H) + a_f(F(\psi(r_H)) - \psi(r_H))\} = \psi^{-1}\{\psi(r_H)\} = r_H,$$

because $\psi(r_H) = n_H$ and $F(n_H) = n_H$. Let us now make a first-order Taylor expansion of $\Psi$ around $r_H$ :

$$\Psi(r^f_t) = \Psi(r_H) + \Psi'(r_H)(r^f_t - r_H) + \varepsilon(r^f_t - r_H) \qquad \text{or}$$

$$\text{(A.1)} \qquad \Psi(r^f_t) - \Psi(r_H) = \Psi'(r_H)(r^f_t - r_H) + \varepsilon(r^f_t - r_H)$$

where $\Psi'(r_H)$ is given by :

$$\Psi'(r_H) = \frac{d(\psi^{-1}(n_H))}{d(n_H)}\left(\psi'(r_H) + a_f(F'(\psi(r_H))\psi'(r_H) - \psi'(r_H)\right)$$

$$\frac{1}{\psi'(r_H)}\left(\psi'(r_H) + a_f(F'(\psi(r_H))\psi'(r_H) - \psi'(r_H)\right)$$

$$= 1 + a_f(F'(\psi(r_H)) - 1) = \theta_f,$$

where $0 < \theta_f < 1$, because $0 < F'(\psi(r_H)) < 1$. This is inserted in (A.1):

$$\Psi(r_t^f) - \Psi(r_H) = \theta_f(r_t^f - r_H) + \varepsilon(r_t^f - r_H) \qquad \text{or}$$

$$\text{(A.3)} \qquad r_{t+1}^f - r_H = \theta_f(r_t^f - r_H) + \varepsilon(r_t^f - r_H),$$

since $r_{t+1}^f = \Psi(r_t^f)$ and $\Psi(r_H) = r_H$. This can be done for the slow learner as well and

$$\text{(A.4)} \qquad r_{t+1}^s - r_H = \theta_s(r_t^s - r_H) + \varepsilon(r_t^s - r_H)$$

where $\theta_s = 1 + a_s(F'(\psi(r_H)) - 1)$ and $0 < \theta_s < 1$. Since $0 < F'(\psi(r_H)) < 1$ and $a_s < a_f$, then $\theta_f < \theta_s$. We can now use the lemma 2.1 from chapter 2 with $x_t = (r_t^s - r_H)$ and $y_t = (r_t^f - r_H)$, and there exists an integer $\tau$ such that

$$(r_t^s - r_H) > (r_t^f - r_H) \qquad \text{for all } t \geq \tau \quad \Rightarrow$$

$$r_t^s > r_t^f \qquad \text{for all } t \geq \tau. \quad \blacksquare$$

*Proof of Proposition 3.3.* This is similar to the proof of proposition 3.2 so we will just briefly skecth the proof. The welfare $U(c_{t+1}) - V(n_t)$ can as the real interest rate be written as a function of the forecast $n_{t+1}^e$:

$$\omega_{t+1} = U(c_{t+1}) - V(n_t) = \xi(n_{t+1}^e).$$

This can be done for both the fast and the slow learner :

$$\omega_{t+1}^f = \xi(n_{t+1}^{f,e}) \quad \text{and} \quad \omega_{t+1}^s = \xi(n_{t+1}^{s,e}).$$

Since $n_{t+1}^{f,e} \to n_H$ when $t \to \infty$ and $n_{t+1}^{s,e} \to n_H$ when $t \to \infty$, the welfare converge as well:

$$\omega_{t+1}^f \to U(c_H) - V(n_H) = \omega_H \qquad \text{and}$$

$$\omega_{t+1}^s \to U(c_H) - V(n_H) = \omega_H \text{ when } t \to \infty.$$

Since $\xi$ is a $C^2$-function with a non-vanishing derivative $\xi'(n_H) \neq 0$, then we can use the inverse function theorem again and there exist a unique $C^1$-function $\zeta$ such that:

$$n_{t+1}^{f,e} = \zeta(\omega_{t+1}^{f}) \quad \text{and} \quad n_{t+1}^{s,e} = \zeta(\omega_{t+1}^{s})$$

for all $(n_{t+1}^{f,e}, \omega_{t+1}^{f}) \in \Omega$ and for all $(n_{t+1}^{s,e}, \omega_{t+1}^{s}) \in \Omega$, where $\Omega$ is a neighbourhood around $(n_H, \omega_H)$.

Let us now look at the first case where $\omega_H > \omega_{t+1}^{f}$ for all t and $\omega_H > \omega_{t+1}^{s}$ for all t. As in the proof of proposition 3.2, we can construct two sequences similar to (A.3) and (A.4) but for the welfare $\omega_{t+1}^{f}$ and $\omega_{t+1}^{s}$ instead of the real interest rate. This is done with the help of the inverse function theorem and by linearising around the steady state $\omega_H$ as done above :

$$\omega_H - \omega_{t+1}^{f} = \theta_f(\omega_H - \omega_t^{f}) + \varepsilon(\omega_H - \omega_t^{f}) \qquad \text{and}$$

$$\omega_H - \omega_{t+1}^{s} = \theta_s(\omega_H - \omega_t^{s}) + \varepsilon(\omega_H - \omega_t^{s}).$$

where $\theta_f$ and $\theta_s$ is as above, hence $\theta_f < \theta_s$ and according to the lemma there exist an integer $\tau$ such that

$$\omega_H - \omega_{t+1}^{f} < \omega_H - \omega_{t+1}^{s} \qquad \text{for all } t \geq \tau \Rightarrow$$

$$\omega_{t+1}^{f} > \omega_{t+1}^{s} \qquad \text{for all } t \geq \tau.$$

The second part of the proposition is similar, but now there exists an integer m such that

$$\omega_{t+1}^{s} > \omega_H \quad \text{for all } t > m, \qquad \text{and}$$

$$\omega_{t+1}^{f} > \omega_H \quad \text{for all } t > m,$$

We construct the following sequences again

$$\omega_{t+1}^{f} - \omega_H = \theta_f(\omega_t^{f} - \omega_H) + \varepsilon(\omega_t^{f} - \omega_H) \qquad \text{and}$$

$$\omega_{t+1}^{s} - \omega_H = \theta_s(\omega_t^{s} - \omega_H) + \varepsilon(\omega_t^{s} - \omega_H).$$

where $\theta_f$ and $\theta_s$ is as above and $\theta_f < \theta_s$ .We can now use lemma 2.1 from chapter 2 such that there there exist an integer $\tau$ and

$$\omega_{t+1}^{f} - \omega_H < \omega_{t+1}^{s} - \omega_H \qquad \text{for all } t \geq \tau \Rightarrow$$

$$\omega_{t+1}^{f} < \omega_{t+1}^{s} \qquad \text{for all } t \geq \tau. \quad \blacksquare$$

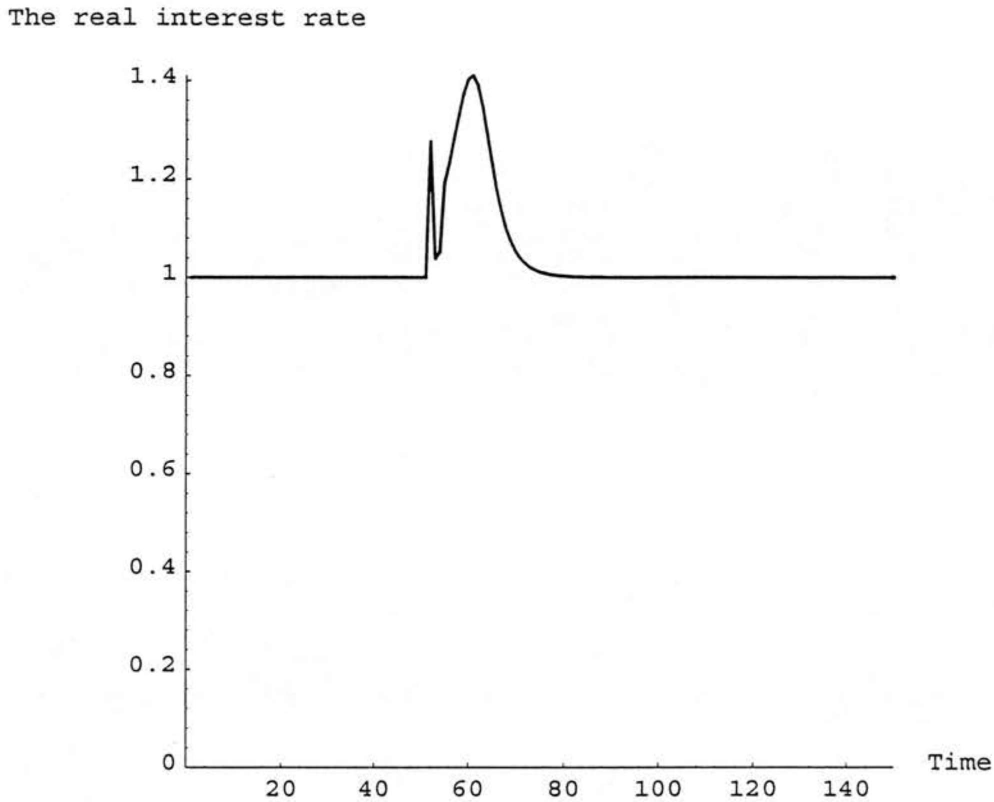


**Figure 3.4. The effect on the real interest rate, when the subsidy is increased from s = 0 to s = 0.25.**

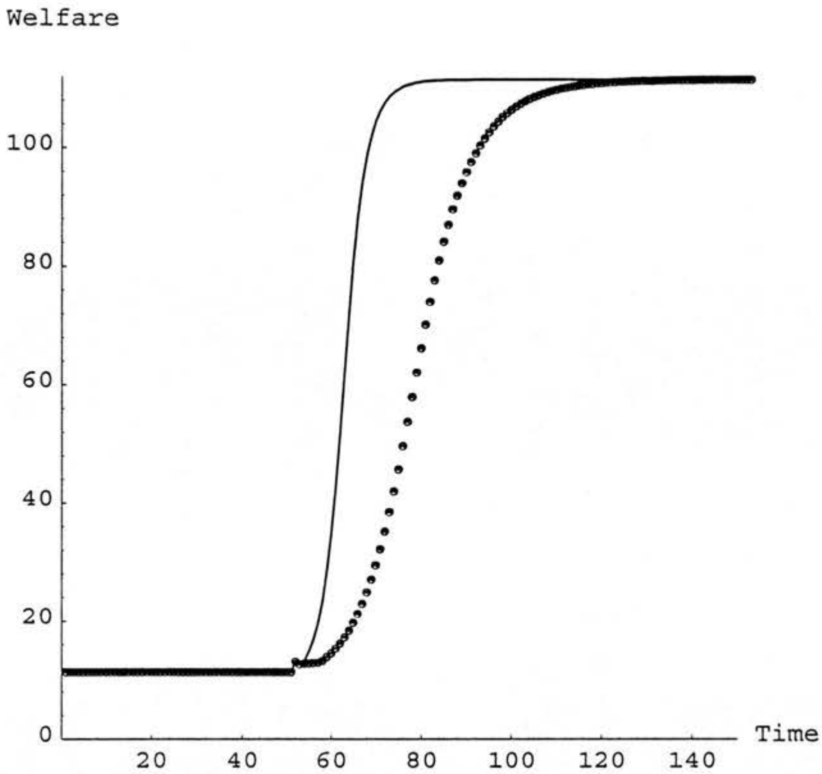


**Figure 3.5 The fast learners welfare (straight line) and the slow learners welfare (dotted line)**

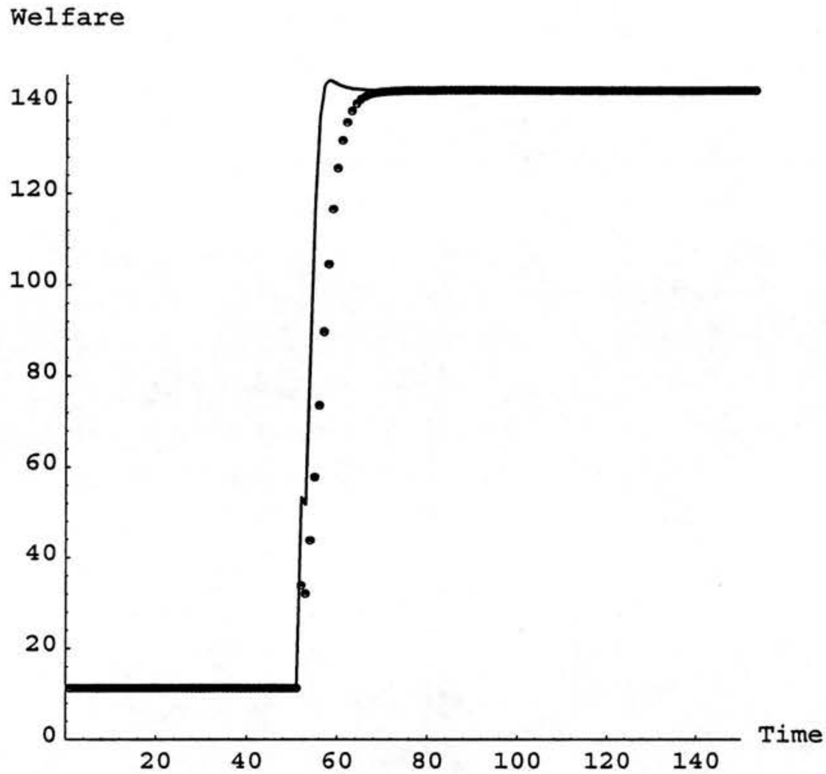


**Figure 3.6 The fast learners welfare (straight line) and the slow learners welfare (dotted line)**

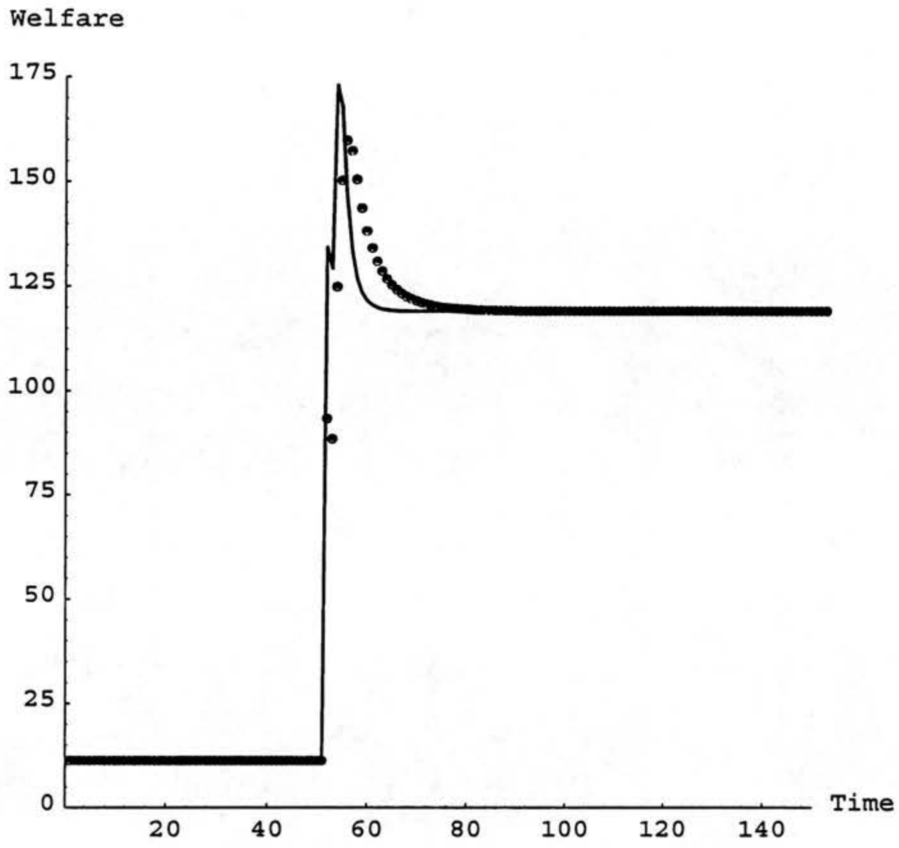


**Figure 3.7 The fast learners welfare (straight line) and the slow learners welfare (dotted line)**

## Chapter 4. The OLG - model with intrinsic noise.

### 4.1. Introduction.

The study of learning in nonlinear models has largely focused on models without intrinsic noise, however the literature on learning in linear models have emphasised a stochastic framework, thus it would be obvious to extend the deterministic nonlinear model from chapter 2 and 3 to a stochastic nonlinear model in order to allow, for example, for a random shock in the production function or a random shock in the welfare function, i.e. a preference shock. These types of shocks are often referred to intrinsic noise.

If there are no random variables in the model decribed chapter 2 and 3 the economy converge to a steady state, as shown in chapter 2 and 3, and the steady states are stable under learning. When we introduce a shock in the model, the stability conditions will depend on the distribution of shock, since the model is nonlinear, and the steady state will depend on the distribution the shock in most cases. It is possible to obtain fluctuations around the steady state if we introduce a random variable. In some simple cases, the steady state does not depend upon the shock. We will briefly analyse the model from chapter 2 with a simple preference shock. The reason for analysing the model from chapter 2 with the preference shock is to investigate, whether the welfare comparison between an economy from chapter 2 with fast learners and an economy with slow learners still holds.

In chapter 3 there were multiple steady states, and we used a subsidy financed by a lump-sum tax to generate fluctuations and shift the economy between steady states. However, we do not need a shift in government policy to generate fluctuations, as long as we have large productivity shocks and a constant speed of learning. Here we will introduce government spending financed by either by a lump-sum tax in addition to the productivity shock. In the previous chapter the agents used a constant speed of learning, however in order to be certain that the steady states are locally stable under learning, we need a decreasing speed of learning. Otherwise a large shock early might push the economy away from the steady state. The main purpose is to analyse the welfare consequences of different speeds of learning as in chapter 3, and see if we get similar results as in chapter 3.

The chapter is organised as follows, in section 4.2 we study the basic model from chapter 2 with a simple preference shock and investigate whether the welfare analysis still hold in this case. In section 4.3, we look at an adaptive learning rule and the conditions for stability of a steady state under learning when we have multiple equilibria and a productivity shock. In section 4.4, we analyse different form of government policies, while in section 4.5 we look at stability issues, when agents use a decreasing speed of learning or a constant speed of learning. In section 4.6, we compare the welfare between two groups of learners when the two groups either have a decreasing speed of learning or a constant speed of learning, and section 4.7 contains the conclusions.

### 4.2. The basic model with a preference shock.

In our basic overlapping generations model the agents live for two periods, where they work when they are young and consume when they are old. An agent born at time t maximise the expected welfare $E_t \{ U(c_{t+1}) - ( V(n_t) - \varepsilon_t n_t )\}$ at time t, where $c_{t+1}$ is consumption at time t + 1, $n_t$ is labour supply at time t, and $e_t$ is an i.i.d. random variable with bounded support, mean 0 and variance $\sigma^2 > 0$. $E_t$ is the expectations held at time t. We make the following assumptions on U, V.

**Assumption 4.1.** U is a strictly concave $C^2$-function with U' > 0 and U'' < 0, V is a strictly convex $C^2$-function with V' > 0 and V'' > 0.

**Assumption 4.2.** $U'(c) \to \infty$ for $c \to 0$ and $V'(n) \to \infty$ for $n \to \infty$.

Assumption 4.2 ensures that the desired labour supply, hence the desired real balances will be positive no matter what the expected real interest rate are. The budget constraints for the households are given by :

$$p_t y_t = M_t^d \qquad \text{and} \qquad p_{t+1} c_{t+1} = M_t^d .$$

where $p_t$ is the price of time t, $p_{t+1}$ is the price at time t + 1, $y_t$ is the output produced at time t and $M_t^d$ is the money demand at time t, which is carried forward to time t + 1. The real interest rate is given by $p_t / p_{t+1}$ . The production function f is a concave $C^2$-function $f(n_t)$ with $f' > 0$ and $f'' \leq 0$.

The introduction of a preference shock follow Woodford (1990), and is a simple way to introduce noise in the model. In this case, the rational expectations equilibria does not depend on the preference shock, it will only be the learning rule that depend on $\varepsilon_t$. There are many other types of preference, technology and monetary shocks, and this give rise to more complicated situations as we shall see below.

Let us return to the households maximisation problem. The first-order condition is given by:

$$\text{(4.1)} \qquad E_t\left( U'\left(\frac{p_t}{p_{t+1}} y_t\right) \cdot \frac{p_t}{p_{t+1}} f'(n_t) + \varepsilon_t \right) = V'(n_t).$$

In the basic model the money supply is often assumed to be constant, such that the equilibrium condition on the money market is $M_t^d = M$ for all t and

$$(M/p_{t+1})/(M/p_t) = y_{t+1}/y_t \qquad \Rightarrow \qquad p_{t+1}/p_t = y_{t+1}/y_t .$$

Since $y_t = f(n_t)$ the real interest rate $p_{t+1}/p_t$ is given by

$$p_{t+1}/p_t = f(n_{t+1})/f(n_t)$$

Hence the first-order condition (4.1) implies

$$\text{(4.2)} \qquad E_t \{ U'(f(n_{t+1}))f(n_{t+1}) + \varepsilon_t \} = \frac{f(n_t)}{f'(n_t)} V'(n_t)$$

$E_t$ denote the agent's expectations at time t, and in a rational expectations equilibrium the agent's expectations are equal to the true conditional distribution at time t and $E_t(\varepsilon_t) = 0$, such that (4.2) is given by

(4.3) $$E_t\{\, U'(f(n_{t+1}))f(n_{t+1})\,\} = \frac{f(n_t)}{f'(n_t)} V'(n_t)\,,$$

thus a rational expectations equilibrium is a stochastic process given by $\{n_t\}_t$ that satisfies (4.3), or equivalently (4.2), since $E_t(\varepsilon_t) = 0$ in a rational expectations equilibrium. Since V is a $C^2$-function with $V'' > 0$ and f is a $C^2$-function with $f'' \leq 0$, the derivative of the right-hand side of (4.2) is positive for all $n > 0$ :

$$\frac{1}{f'(n)^2}\{\, (V''(n)f(n) + V'(n)f'(n))f'(n) - f''(n)f'(n)V'(n)\,\} > 0,$$

and the right-hand side of (4.2) has an inverse function, and we can rearrange (4.2) so the labour supply at time t, $n_t$, depends on the labour supply at time t + 1, $n_{t+1}$ :

(4.4) $$n_t = H\{\, E_t(U'(f(n_{t+1}))f(n_{t+1})) + \varepsilon_t)\,\} = F(n_{t+1}, \varepsilon_t)$$

where H is the inverse of $(f(n)/f'(n))V'(n)$.

The process given by $n_t = \bar{n} > 0$ for all t, where $\bar{n}$ is a solution to $U'(\bar{n}) = (1/f'(\bar{n}))V'(\bar{n})$ is the well-known steady state where money has value. Assumption 4.1 and 4.2 ensures the existence and uniqueness of $\bar{n} > 0$, see Woodford (1990). If $\lim_{c\to\infty} U'(c)c = 0$, then $n_t = 0$ for all t is also a solution, and this is the autarchy steady state where money does not have any value. The nonstochastic version of (4.4) can have different types of rational expectations equilibria, for example, periodic equilibria, and sunspot solutions. Azariadis (1981), shows that there also exists stationary rational expectations equilibria where prices and labour supply are stochastic without any form of random shocks in the model. Azariadis (1981) and Woodford (1990) look extensively at sunspot equilibria in this model[1] and find conditions for the existence of sunspot equilibria.

We wish to consider when the monetary steady state is locally stable under learning. The agents use the adaptive learning rule from chapter 2 and chapter 3 :

(4.5) $$n^e_{t+1} = n^e_t + a_t(n_{t-1} - n^e_t) \qquad \text{for all t,}$$

[1] A sunspot equilibrium is a rational expectations equilibrium where purely extrinsic uncertainty affect the equilibrium prices, labour supply and consumption.

where $0 < a_t \leq 1$ and $\sum_{t=2}^{\infty} a_t = \infty$. In order to obtain local stability, we need the following assumption on the speed of learning $a_t$.

**Assumption 4.3.** $a_t = a(1/t)$ for all t, where $0 < a \leq 1$.

Actually we just need certain conditions on the rate of $a_t \to 0$ when $t \to \infty$, but assumption 4.4 satisfies these conditions, see Evans and Honkapohja (1995a). Hence we have a decreasing speed of learning, this is sometimes called a decreasing gain parameter. If $a_t$ was constant, it is possible that a large shock early might push the economy away from the steady state.

The local stability of the monetary steady state under learning is described in Woodford and Evans and Honkapohja (1995a). We have to distinguish between weak and strong stability as described below, because the stability conditions are sensitive to a variation in the perceptions which agents are assumed to hold. If the agents use additional variables in their forecasting rule, for example, if they believe that the economy is in a k-cycle or a sunspot equilibrium this can alter the stability conditions. Woodford shows that if agents include an additional variable in their forecasting rule, the monetary steady state might be unstable, while it is stable if this additional variable is omitted. Evans and Honkapohja (1995a) gives an example of a steady state that is weakly stable but not strongly stable.

Let us use the approach from Evans and Honkapohja (1995a), where the local stability of a steady state under a learning rule like (4.5) is analysed, and it is shown that local stability under learning is determined by expectational stability conditions. The expectational stability or E-stability conditions are developed as follows. The agents forecast at time t the labour supply at time t + 1, where the forecast is given by $n^e_{t+1}$ according to (4.5). We can combine this with (4.4) and $n^e_{t+1}$ is given by

$$n^e_{t+1} = n^e_t + a_t(F(n^e_t, \varepsilon_{t-1}) - n^e_t)$$

Given this perceived value of the labour supply the actual value of the labour supply is given by (4.4) according to

$$n_t = F(n_{t+1}^e, \varepsilon_t).$$

Hence we have a mapping T: R $\rightarrow$ R from the perceived value to the actual value given by

$$T(n) = F(n,\varepsilon),$$

where $\varepsilon$ is a random variable with the same distribution as $\varepsilon_t$. The steady state $\bar{n}$ is a fixed point of T and the definition of E-stability is based on the differential equation

(4.6) $$dn/d\tau = T(n(\tau)) - n(\tau),$$

where $\tau$ is notional time. A fixed point $\bar{n}$ of T is E-stable if (4.6) is locally stable at $\bar{n}$. If $\bar{n}$ is not E-stable then it is E-unstable. However E-stability is defined relative to the perceptions the agents hold, and this leads to a definition of weak E-stability and strong E-stability as mentioned above. The steady state is weakly E-stable if it is stable under the learning algorithm (4.5), and strongly E-stable if it is stable under an overparameterized learning rule, for example, if there was an extra lag in (4.5) or the agents believed the economy could be in a cycle, and formulated a learning rule according to these beliefs.

The necessary and sufficient condition for weak and strong E-stability of the steady state $\bar{n}$ is given in proposition 3.1 in Evans and Honkapohja (1995a).

- $\bar{n}$ is weakly E-stable if and only if $F'(\bar{n},\varepsilon) < 1$.
- $\bar{n}$ is strongly E-stable if and only if $| F'(\bar{n},\varepsilon) | < 1$.

Under appropriate conditions stated in Evans and Honkapohja (1995a), the local convergence of $n_t^e$ to the steady state $\bar{n}$ is governed by the stability of (4.6), i.e. the E-stability conditions. Proposition 5.2 from Evans and Honkapohja (1995a) gives a condition for local stability under learning, if we have "small" shocks.

- If the initial point $n_0^e$ is sufficiently close to $\bar{n}$ and $\bar{n}$ is weakly E-stable, then $n_t^e \rightarrow \bar{n}$ with probability 1.

A similar result hold for strong E-stability. The reason for having "small" shocks is to avoid a projection facility described in Evans and Honkapohja (1995a). Let us assume there are no limits on the size of the shocks. If we have a large shock early on, it might push the learning process away from a neighbourhood around the steady state. Thus the learning rule does not converge with probability 1. In order to avoid this possibility, Evans and Honkapohja introduce a projection facility to keep the learning algorithm in a neighbourhood of $\bar{n}$, when shock does not have a small support. This projection facility is also used in Marcet and Sargent (1989b). Grandmont and Laroque (1991) give a critical review of the projection facility.

We have focused on local stability of the steady state, the reason for this is that there are not many results on global stability of the steady state under learning. There is a global convergence result in Evans and Honkapohja (1995d), but we have to find a Lyapounov function in order to use the result.

With these stability results in mind, let us turn to the welfare comparison between an economy with fast learners and an economy with slow learners, where the fast agents speed of learning is given by $a_t^f = a_f(1/t)$ for all t and the slow is given by $a_t^s = a_s(1/t)$ and $1 \geq a_f > a_s > 0$. We consider two situations, one where both groups start above the steady state and one where both groups start below the steady state. These case corresponds to the situations analysed in section 2.6 of chapter 2. As in chapter 2, we use the following example.

**Example 4.1.** Let $U(c) = (1/(1-\sigma))c^{1-\sigma}$, $V(n) = (1/(1+\varepsilon))n^{1+\varepsilon}$ and $f(n) = An^{\alpha}$ where $0 < \sigma < 1$, $\varepsilon > 0$, $A > 0$ and $0 < \alpha < 1$.

Let us assume that $n_1^{f,e} = n_1^{s,e} = n_1^e$ is *above* $\bar{n}$, and let $\{n_{t+1}^{f,e}\}_t$ denote the fast learners sequence of expected labour supply and $\{n_{t+1}^{s,e}\}_t$ denote the slow learners sequence of expected labour supply. We calculate the actual labour supply and consumption for the two groups, where $\{n_t^f, c_{t+1}^f\}_t$ denote the fast learners labour supply and consumption and $\{n_t^s, c_{t+1}^s\}_t$ are the slow learners labour supply and consumption. This is used to calculate the welfare for the two groups and make a comparison between the level of welfare of a fast learner born at time t and the welfare of a slow learner born at time t as in chapter 2. We have the following figure for the welfare during the learning transition to the monetary steady state $\bar{n}$.

Figure 4.1 about here. [2]

The initial condition is $n_1^e$ =10 and $\bar{n}$ = 1.3907. We have the same picture as figure 2.5 in chapter 2, the slow learners are better off initially compared to the fast learners, but the economy is close to the steady state the fast learners overtake the slow learners.

Let us assume that $n_0^{f,e} = n_0^{s,e} = n_0^e$ is *less than* $\bar{n}$, so that we start below the steady state. In this case the welfare analysis reverse the result from before, the fast learners are better off initially, but the slow learners are better off when the economy is close to the steady state, as figure 4.2a show.

Figure 4.2a about here.

The parameter values are the same as in figure 4.1, but $a_f = 0.9$ and $a_s = 0.5$ with the initial condition $n_0^e$ = 0.5. However we can have a different figure if we change $a_f$ and $a_s$ to $a_f = 0.99$ and $a_s = 0.2$. These parameters give the following figure.

Figure 4.2b about here.

Here the slow learner is better off when $3 \leq t \leq 6$, but for $t \leq 2$ and $t \geq 7$ the fast learners are better off. In this case there are two crossing points and the welfare results from chapter 2 is not valid in this case. A reason for the two crossing points might be the low value of $a_s = 0.2$ such that the speed of learning $a_t^s = a_s(1/t) = 0.2/t$

[2] We use the following parameter values: $\alpha = 0.2$, $A = 28$, $\sigma = 0.41$, and $\varepsilon = 0.2$, with $a_f = 0.5$, $a_s = 0.2$ and $\varepsilon_t$ is uniform distributed over the interval [ −0.1, 0.1 ]

is low and convergence will be slow. The slow learners does produce very good forecasts compared to fast learners, and the slow learners welfare actually fall below the steady state level in figure 4.2b. This is not clearly shown in the figure, but this due to the scaling.

If we choose the parameters careful, we can get two crossing points. It is also possible to find parameter values, where the fast learners are better off during the entire learning transition, and we cannot use the welfare results from chapter 2 in this case.

### 4.3. Increasing social returns and different forms for intrinsic noise.

Let us now use the production function from chapter 3 in order to investigate whether the welfare comparison from chapter 3 still holds if we introduce a random variable. The welfare analysis from chapter 3 might be used, if we use a productivity shock since we have increasing returns in the production function. The production function is given by :

(4.7) $$f(n_t, Kn_t) = A(n_t)^{\alpha}\{\max(I^*, \lambda Kn_t(1 + b\lambda Kn_t)^{-1}\}^{\beta}.$$

Furthermore, let us incorporate a random shock in the model in more complicated way than above, this might be a preference shock or a productivity shock. We consider a productivity shock $v_t$, where $v_t$ is observed at time t before the agent has decided on labour supply etc. This is incorporated as follows :

(4.8) $$y_t = f(n_t, Kn_t)v_t$$

where $v_t$ is a positive i.i.d. shock with bounded support, mean $E(v_t) = 1$, and variance $var(v_t) = \sigma^2$.

Instead of a productivity shock we could introduce a preference shock $\varepsilon_t$, as before but in a more general form :

$$U(c_{t+1}) - V(n_t) + \varepsilon_t \ln(n_t),$$

where $\varepsilon_t$ is an i.i.d. stochastic variable with bounded support, mean 0 and variance $\sigma^2$ > 0. In this case the steady state depends on the distribution of the preference shock. There are of course many other types of preference, technology and monetary shocks we could introduce in the model, but in the following we will focus the attention on a productivity shock like (4.8), where $v_t$ is given by :

$$v_t = 1 + \nu(0.5 - u_t),$$

and $u_t$ is i.i.d. uniform over the unit interval with $\nu = 0.20$.

Let us return to the household's maximisation problem. The first-order condition is given now by:

$$E_t\left( U'\left(\frac{p_t}{p_{t+1}} y_t\right) \cdot \frac{p_t}{p_{t+1}} f_1(n_t, Kn_t) v_t \right) = V'(n_t) \qquad \Rightarrow$$

$$(4.9) \qquad E_t\left( U'\left(\frac{p_t}{p_{t+1}} f(n_t, Kn_t) v_t\right) \cdot \frac{p_t}{p_{t+1}} v_t \right) = f_1(n_t, Kn_t) V'(n_t),$$

where $f_1$ is the partial derivative of f w.r.t its first argument. In the basic model the money supply was constant, such that the equilibrium condition on the money market is $M_t^d = M$ for all t and

$$(M/p_{t+1})/(M/p_t) = y_{t+1}/y_t \qquad \text{or} \qquad p_{t+1}/p_t = y_{t+1}/y_t\,.$$

Since $y_t = f(n_t, Kn_t)v_t$, the real interest rate $p_{t+1}/p_t$ is given by

$$p_{t+1}/p_t = f(n_{t+1}, Kn_{t+1})v_{t+1}/f(n_t, Kn_t)v_t$$

In a rational expectations equilibrium the agent's expectations about the distribution of $p_{t+1}/p_t$ will be equal to the true conditional distribution for $f(n_{t+1}, Kn_{t+1})v_{t+1}/f(n_t, Kn_t)v_t$, and the first-order condition (4.9) becomes

$$E_t\{ U'(f(n_{t+1}, Kn_{t+1})v_{t+1}) \} f(n_{t+1}, Kn_{t+1})v_{t+1} = \frac{f(n_t, Kn_t)}{f_1(n_t, Kn_t)} V'(n_t)$$

This is the temporary equilibrium condition, and can be written in the reduced form where the labour supply at time t, $n_t$ depends on the labour supply at time t + 1, $n_{t+1}$:

$$(4.10) \qquad n_t = H\{ E_t G(n_{t+1}, v_{t+1}) \}$$

with $G(n_{t+1}, v_{t+1}) = E_t\{U'(f(n_{t+1}, Kn_{t+1})v_{t+1})\} f(n_{t+1}, Kn_{t+1})v_{t+1}$. In this case the reduced form does not depend on $v_t$. If we used $y_t = f(n_t, Kn_t) + v_t$ then $n_t$ would also depend on $v_t$, and the reduced form is given by

$$(4.11) \qquad n_t = H\{ E_t G(n_{t+1}, v_{t+1}), v_t \}$$

A steady state in the stochastic case is defined as follows.

**Definition 4.1.** A rational expectations equilibrium steady state is a solution $n_t = \bar{n}(v_t)$ where

$$\bar{n}(v_t) = H(E_w(G(\bar{n}(w),w),v_t).$$

$E_w$ denotes the expectation over the random variable w, and where w has the same distribution as the i.i.d. shocks $v_t$. In definition 4.1, the steady state depends on the random shock $v_t$, but a steady state for the reduced form in (4.10) does not depend on $v_t$. In this case the rational expectations equilibrium steady state is defined as

$$\bar{n} = H(E_w(G(\bar{n},w)).$$

### 4.4. The government policy.

We will again study government spending under different forms of financing. The government buys a constant proportion of output, $0 \leq \gamma < 1$ at each time t, such that

(4.12) $\quad g_t = \gamma f(n_t, Kn_t)$,

where $g_t$ is government spending at time t. We assume that the government spending $g_t$ is distributed back to the household through a non-decreasing function Z included in the welfare function, $U(c_{t+1}) - V(n_t) + Z(g_{t+1})$. The purchase $g_t$ is either financed entirely by a lump-sum tax with the money supply M kept constant or financed entirely by money creation, it could also be a mixture of both. Hence the government's budget constraint is either :

(4.13) $\quad T_t = p_t g_t \quad$ or

(4.14) $\quad M_t^s = M_{t-1}^s + p_t g_t$

where $T_t$ is the lump-sum tax at time t and $M_t^s$ is the money supply at time t.

If we begin with spending financed by a lump-sum tax, the households pay the tax $T_t$ at time t and the household's budget constraint at time t is changed to $p_t y_t - T_t = M_t^d$. Since the money supply is constant and inserting (4.12) and (4.13) into the households budget-constraint, we have $(1-\gamma)p_t y_t = M$, and

$$\frac{M / p_{t+1}}{M / p_t} = \frac{f(n_{t+1}, Kn_{t+1})v_{t+1}}{f(n_t, Kn_t)v_t} \qquad \Rightarrow$$

(4.15) $$\frac{p_t}{p_{t+1}} = \frac{f(n_{t+1}, Kn_{t+1})v_{t+1}}{f(n_t, Kn_t)v_t} \quad \text{for all t.}$$

where $p_t / p_{t+1}$ is the real interest rate at time t + 1. The first-order condition for the households problem is then given by :

(4.16) $$E_t\left( U'\left(\frac{1}{p_{t+1}}\{p_t y_t - T_t\}\right) \cdot \frac{p_t}{p_{t+1}} f_1(n_t, Kn_t)v_t \right) = V'(n_t) \qquad \Rightarrow$$

where $E_t$ is the expectations held at time t and $f_1$ is the partial derivative w.r.t. first argument $n_t$. If we substitute (4.12) and (4.13) into (4.16), we have :

(4.17) $$E_t\left( U'\left(\frac{p_t}{p_{t+1}}(1-\gamma)f(n_t, Kn_t)v_t\right) \cdot \frac{p_t}{p_{t+1}} f_1(n_t, Kn_t)v_t \right) = V'(n_t).$$

Insert (4.15) into (4.17) such that :

$$E_t\left( U'\{(1-\gamma)f(n_{t+1}, Kn_{t+1})v_{t+1}\} \cdot \frac{f(n_{t+1}, Kn_{t+1})v_{t+1}}{f(n_t, Kn_t)} f_1(n_t, Kn_t) \right) = V'(n_t)$$

This can be rearranged to obtain :

$$E_t\left( U'\left((1-\gamma)f(n_{t+1}, Kn_{t+1})v_{t+1}\right) f(n_{t+1}, Kn_{t+1})v_{t+1} \right) = \frac{f(n_t, Kn_t)}{f_1(n_t, Kn_t)} V'(n_t).$$

Given the production function f with increasing returns then :

$$\frac{f(n_t, Kn_t)}{f_1(n_t, Kn_t)} = \alpha n_t .$$

Since $V'(n_t)$ is a $C^1$-function with $V'' > 0$, we can write $n_t$ as a function of $n_{t+1}$, $v_{t+1}$, and $\gamma$ in the reduced form :

(4.18) $$n_t = H(E_t(G(n_{t+1}, v_{t+1}, \gamma)))$$

where $G(n_{t+1}, v_{t+1}, \gamma) = (U'((1-\gamma)f(n_{t+1}, Kn_{t+1})v_{t+1}) \cdot f(n_{t+1}, Kn_{t+1})v_{t+1})$. This is similar to the reduced form (4.10) apart from $\gamma$.

**Example 4.2.** We use the functions from example 4.1 but with the production function given in (4.7), to find an explicit expression for $H(E_t(G(n,v,\gamma))$ :

$$G(n_{t+1}, v_{t+1}, \gamma) = (1-\gamma)^{-\sigma}\{A(n_{t+1})^{\alpha}\{\max(I^*, \lambda Kn_{t+1}(1 + b\lambda Kn_{t+1})^{-1}\}^{\beta} v_{t+1}\}^{1-\sigma}$$

and $n_t = H(E_t(G(n_{t+1}, v_{t+1}, \gamma))) = \left(\alpha E_t\left(G(n_{t+1}, v_{t+1}, \gamma)\right)\right)^{\frac{1}{1+\varepsilon}}$ and

(4.19) $$H(E_t(G(n_{t+1}, v_{t+1}, \gamma)))$$

$$= \left(\alpha\left(\frac{1}{1-\gamma}\right)^{\sigma}\right)^{\frac{1}{1+\varepsilon}} E_t \left( A(n_{t+1})^{\alpha} \left\{ \max(I^*, \frac{\lambda K n_{t+1}}{1+b\lambda K n_{t+1}}) \right\}^{\beta} v_{t+1} \right)^{\frac{1-\sigma}{1+\varepsilon}}$$

where $0 < \alpha < 1$, $A > 0$, $\beta > 1$, $\lambda > 0$, $\gamma \geq 0$, $I^* > 0$, $b > 0$, and $K > 0$.

Let us briefly study the various cases when we have multiple equilibria and possible coordination failures. This is simpler to understand in a non-stochastic set-up as in chapter 3, hence $y_t = f(n_t, Kn_t)$ and if the agents have perfect foresight then $n_t$ is a function of $n_{t+1}$ :

$$n_t = H(G(n_{t+1}, \gamma)) = F(n_{t+1}, \gamma).$$

We are still analysing the situation where we assume that the substitution-effect dominates the income-effect, such that F is upward sloping as shown in figure 4.3. For different values of $\gamma$ there can be 0,1,2, and 3 interior steady states, denoted $n_L$, $n_U$ and $n_H$. In figure 4.3 we have illustrated F for different values of $\gamma$ and there can be 1,2, and 3 interior steady states depending on the size of $\gamma$. This is also the case when we have a productivity shock.

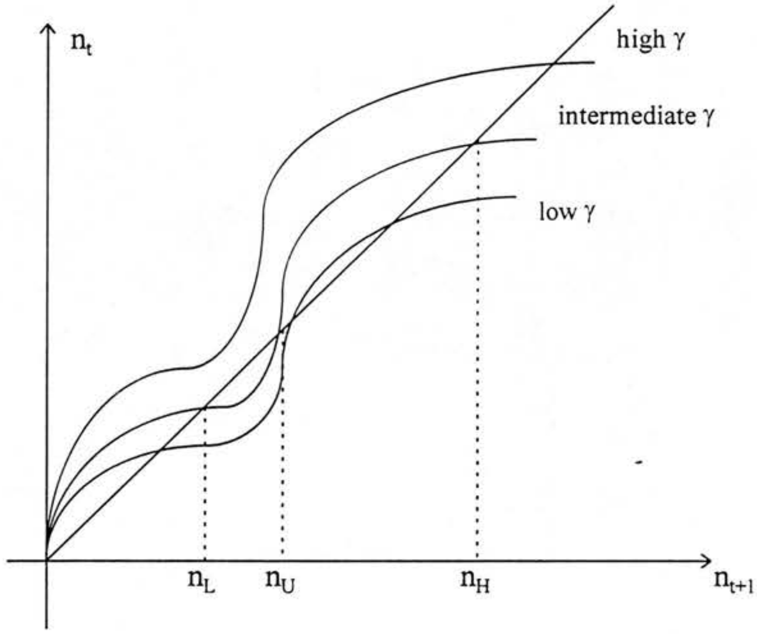


**Figure 4.3.** The F-function for different values of $\gamma$. [3]

[3] We use the following parameter values : $\alpha = 0.9$, $A = 0.0805$, $\beta = 1.007\ 1$, $\lambda = 0.5$, $I^* = 19.5$, $K = 40$, $\sigma = 0.1$, $b = 0.025$ and $\varepsilon = 0.25$.

The interior steady states correspond to "noisy" steady states given by

$$n_t = \bar{n}_i \text{ and } y_t = f(\bar{n}_i, K\bar{n}_i)v_t, \text{ for } i = L,U,H,$$

if the size of the shock is "small", i.e. the support of $v_t$ is "small", as shown in Evans and Honkapohja (1995a). The steady state does not depend on the shock $v_t$ in this particular case. We can rank the steady states for given $\gamma$ as we did in chapter 3, such that the welfare at $n_L$ is less than the welfare at $n_U$, and this is less than welfare at $n_H$, if the utility from government spending $Z(g_{t+1})$ is zero or sufficiently low. At the low-level equilibria $n_L$ or $n_U$ agents work less hard than at $n_H$, because all the other agents work less hard. If agents could coordinate their effort to a high level $n_H$ they would better off in terms of welfare.

There is a role for government policy as in chapter 3. If we initially have three interior steady states, $n_L$, $n_U$, and $n_H$ and the economy is stuck in the low-level steady state $n_L$, the government could increase the proportion $\gamma$ to shift the economy to a high level steady state. This increase would rotate the F-function upwards, such that there is only one interior steady state, $n_H$, if the increase in $\gamma$ is large enough. The households supply more labour since the real interest rate $p_t/p_{t+1}$ is above 1, but decreases and converges to 1, hence the households get a high price today for its output. Since the price tomorrow is less than today, they work harder in order to substitute between expensive goods today and cheap goods tomorrow, because we have assumed that the substitution effect dominates the income-effect.

If the government instead use seignorage to finance spending according to

$$M_t^s = M_{t-1}^s + p_t g_t,$$

then the F-function rotates downwards when $\gamma$ is increased and "$\gamma$ low" would be switched to "$\gamma$ high" in figure 4.3. If there are three interior steady states as in figure 4.1, and the economy are initially at $n_L$, the increase in $\gamma$ financed by an increase in the money supply, would lead to a lower labour supply in the steady state. The increase in money supply leads to higher inflation, because the real interest rate $p_t/p_{t+1}$ is less than 1 and increasing. The rise in inflation give a lower return on the work effort, and since we have assumed that the substitution-effect dominates the income-effect, the labour supply decreases for higher levels of $\gamma$. This situation is analysed extensively in Evans and Honkapohja (1993).

The changes in $\gamma$ can be seen as fluctuations caused by productivity shocks if the agents have poor information about changes in $\gamma$, and the data on $\gamma$ are infrequent. It is thus possible for the government to change $\gamma$ more often without the agents incorporating changes in $\gamma$ into their expectations. The case with the production subsidy, which we studied in chapter 2 and in chapter 3 will not be considered here, since it is similar to government spending financed by a lump-sum tax.

## 4.5. The learning rule.

It might seem strange to use a pure rational expectations approach when there is a possibility of multiple rational expectations equilibria, because the approach does not give us a selection criteria for which of the equilibria we should choose. Here we are back to the conjecture by Lucas (1986), that a reasonable learning rule should pick out the relevant equilibrium. Thus in order to overcome this problem with multiple equilibria, we use a learning rule to formulate a forecast of the economic variables of importance and replace the rational expectations assumptions with an adaptive learning rule. Hence the reduced form (4.18) is written as follows:

(4.20) $$n_t = H(\, G(n_{t+1}, v_{t+1}, \gamma)^e \,)$$

where the superscript **e** denote the expectations of $G(n_{t+1}, v_{t+1}, \gamma)$ at time t. The expectations of $G(n_{t+1}, v_{t+1}, \gamma)$ is based on the learning rule, and the actual evolution of the economy depends on the sequence of stochastic shocks $(v_t)_t$ as well as the learning rule and the initial conditions. The issue of interest, is the convergence of the learning algorithm to a rational steady state, but this is not as straightforward as we might expect, since the agents change their forecasts when the dynamic model evolves over time and the evolution of the model depends on the forecasting rule the agents use as well as the sequence of shocks.

The adaptive learning rule is as before, but the expectations are formulated for $G(n_{t+1}, v_{t+1}, \gamma)$:

(4.21) $\theta_{t+1} = \theta_t + a_t(G(n_{t-1}, v_{t-1}, \gamma) - \theta_t)$ for all t,

where $\theta_t = G(n_{t+1}, v_{t+1}, \gamma)^e$ for all t. We assume that $0 < a_t \leq 1$ with $\sum_{t=2}^{\infty} a_t = \infty$, and $a_t$ satisfies assumption 4.3, such that we have a decreasing speed of learning.

In the situation with three rational steady states $n_L$, $n_U$, and $n_H$ in the deterministic case, $n_L$ and $n_H$ where stable under the learning rule (4.21) as shown in chapter 3. This can be generalised to stochastic case, thus $n_L$ and $n_H$ are stable under learning (4.21) in the stochastic case. The argument is based on E-stability conditions developed in Evans and Honkapohja (1995a).

At time t the agent make a forecast of $G(n_{t+1}, v_{t+1}, \gamma)$. Let $\theta_{t+1} = G(n_{t+1}, v_{t+1}, \gamma)^e$ be the forecast obtained from the perceived law of motion, then the actual law of motion is given by $n_t = H(\theta_{t+1})$. The corresponding parameters of the actual law of motion induced by the perceived law of motion is thus given by $E_{v_t}(G(n_t, v_t, \gamma))$. Hence we have a mapping $T : R \to R$ from the perceived law of motion to the actual law of motion given by

$$T(\theta) = E_v(G(H(\theta), v, \gamma))$$

A steady state $\theta^*$ is just a fixed point of T. The local stability of a steady state under learning is equivalent to conditions for E-stability, this is defined as follows.

**Definition 4.2.** A fixed point $\theta^*$ of the mapping T is E-stable (expectational stable) if the differential equation given by

$$d\theta/d\tau = T(\theta(\tau)) - \theta(\tau)$$

is locally stable at $\theta^*$.

*However as mentioned earlier, the definition of E-stability depends on the perceptions which agents are assumed to hold.* This leads to a distinction between weak E-stability and strong E-stability. The steady state is weakly E-stable if it is locally stable under the learning algorithm used above, while it is strongly E-stable if the steady state is locally stable under learning even if the agents overparametrize their learning rule, i.e. it depends on irrelevant variables. Hence if the agents believed in other solutions than the steady states, and formulated a learning rule according to these perceptions, then the steady state is strongly E-stable if it locally stable under this new learning rule.

Let us assume that $v_t$ is a random shock with a small enough support, i.e. $\alpha \geq 0$ is sufficiently "small", such that we can use propositions 5.1 and 5.2 from Evans and Honkapohja (1995a) to find the necessary and sufficient conditions for weak and strong E-stability of the noisy steady states. If $n_L$ , $n_U$ , and $n_H$ denote the three interior steady states from the deterministic case, then there exists "noisy" rational steady states denoted by $\bar{n}_L$, $\bar{n}_U$, and $\bar{n}_H$ according to proposition 5.1, where each these is located in a neighbourhood around the respective deterministic steady state :

$\forall\delta > 0$, there exists an $\alpha^*$ such that $\forall 0 < \alpha < \alpha^*$ there exists a noisy steady state $\bar{n}_i(\alpha)$, for $i = L,U,H$, where $|n - \bar{n}_i(\alpha)| < \delta$ for $i = L,U,H$.

Furthermore $\bar{n}_i(\alpha)$, for $i = L,U,H$, is weakly E-stable , respectively strongly E-stable, for all $\alpha$ sufficiently small if and only if $n_i$ for $i = L,U,H$ , is weakly E-stable, respectively strongly E-stable.

From proposition 5.2, we have for all $\alpha \geq 0$ sufficiently small and for all initial points $\theta_0$ sufficient close to $\bar{n}_i$, that the learning rule (4.21) converge with probability 1 to the noisy steady state $\bar{n}_i$, if the deterministic steady state $n_i$ is weakly stable. A similar result can be shown for strong E-stability. We have from chapter 3 that only $n_L$ and $n_H$ are weakly E-stable, thus only $\bar{n}_L$ and $\bar{n}_H$ are locally stable under learning. Hence the stability under learning from the deterministic case in chapter 3 can be extended to stochastic case.

The convergence to either $n_L$ or $n_H$ depends on the initial conditions and the distribution of shocks $v_t$ , such that the nonlinear stochastic model exhibits path dependence. If the random shock has a "large" support the noisy steady state is only weakly E-stable and not necessarily strongly E-stable, as shown in an example in Evans and Honkapohja (1995a). Hence if the agents overparameterize their learning rule such that it depends on "irrelevant" variables, for example an extra lag in the learning rule (4.21), $n_{t-2} - \theta^e_{t-1}$, then the stochastic steady state is only weakly E-stable but not strongly E-stable under this alternative learning rule.

In the learning rule (4.21) we assumed that the speed of learning $a_t$ was decreasing, let us briefly look at the case where the speed of learning $a_t$ is constant, $a_t = a$. This is analysed in greater detail in Evans and Honkapohja (1993). A decreasing speed of learning is appropriate, if the agents believe that they are in an economy where the variable being forecasted, $\theta_t = G(n_{t+1}, v_{t+1}, \gamma)^e$, has a constant mean over time. This would be reasonable if the agents assume that the structure of the economy never changes, but such an assumption does not seem realistic in practice. If the agents believe that the structure of the economy may be subject to changes, the agents could use a constant speed of learning, and the learning rule will adapt more rapidly to changes in the economy. However a constant speed of learning produce more noisy forecasts, but changes can be adjusted more swiftly.

Evans and Honkapohja (1993) analyse this and show that a constant speed of learning is clearly in better in than a decreasing speed of learning when the spending parameter $\gamma_t$ varies systematically over time. Given a set of parameter-values they can find a constant "equilibrium" speed of learning or an "equilibrium" learning rule, in the sense that the choice of $\boldsymbol{a}$ minimises the mean square forecast error for each agent, given that all the other agents use the speed of learning $\boldsymbol{a}$. Not surprisingly the "equilibrium" speed of learning depends upon the policy parameter.

The introduction of productivity shocks can lead to shifts between high and low activity levels, even there is no shift in the policy or any structural changes, when the agents use a constant speed of learning, because their forecast still generate some randomness in the limit through $a_t\, G(n_{t-1}, v_{t-1}, \gamma)$. If the economy starts close to a low level equilibrium $n_L$, then a large shock in the right direction could lead to large revision in the forecast $\theta_{t+1}$ and move the economy to the high level steady state as shown in figure 4.4 below.

figure 4.4 about here. [4]

The system seems to be alternating between the high level steady state and the low level steady state according to the figure, hence a sequence of large shocks can apparently induce a self-fulfilling overreaction such that the economy is moving between two stable steady states.

Evans and Honkapohja discuss why the agents should believe in a constant speed of learning if the situation is as in figure 4.4, basically there are two reasons. First, the agents may be aware of the possibility of structural changes or policy shifts, and thus use a constant speed of learning. The other reason is that the constant speed of learning may be an "equilibrium" speed of learning, but one problem might be that ***a*** depend on the chosen parameter values and there is no proof of existence of such an equilibrium.

The economy can converge to a stable steady state when there are multiple steady states as shown below.

figure 4.5 about here. [5]

[4] The parameter values are: $\alpha = 0.9$, $A = 0.0805$, $\beta = 1.007\ 1$, $\lambda = 0.5$, $I^* = 19.5$, $K = 40$, $\sigma = 0.1$, $b = 0.025$ and $\varepsilon = 0.25$. The speed of learning is $a = 0.15$ and the steady states are $n_H = 2.2$ and $n_L = 1.8$, $\gamma = 0$.

[5] We use the following parameter values : $\alpha = 0.2$, $A = 1$, $\beta = 4.8\ 1$, $\lambda = 0.01$, $I^* = 2$, $K = 200$, $\sigma = 0.41$, $b = 0.2$ and $\varepsilon = 0.2$. The speed of learning $a = 0.8$ and $n_H = 6.3$.

Figure 4.5 illustrates the "noisy" steady state quite well. It might be interesting to investigate whether the results from chapter 3 still holds when we use the constant speed of learning. We can compare the two situations, a decreasing speed of learning and a constant speed of learning, as noted above the constant speed of learning give a more rapid adjustment but also more noisy forecast. If the economy converge to a stable steady state, we might not be able to distinguish between fast and slow learners, since the agents produce a noisy forecast in the limit.

### 4.6. The welfare comparison between an economy with fast learners and an economy with slow learners.

We will analyse the case where the economy are in a low-level equilibrium $n_L$ with $\gamma = 0$, then the government increase spending from $\gamma = 0$ to $\gamma > 0$ such that the increase is large enough to remove $n_L$ and $n_U$. The welfare comparison is similar to chapter 2 and 3, we have two groups characterised by the speed of learning such that the fast agents are given by $a_t^f = a_f(1/t)$ for all t and the slow agents are given by $a_t^s = a_s(1/t)$ for all t, with $a_f > a_s$. In figure 4.6 we have the welfare comparison, where $\gamma$ is increased at time t = 50 from 0 to 0.25 with the parameter values from footnote 5, but the speed of learning $a_f = 0.99$ and $a_s = 0.8$. We choose the high values in order to speed up convergence.

Figure 4.6 about here.

At time t = 50 the old generation benefits from government spending, since it works more like a subsidy and the old generation does not have to pay the tax at t = 50. However the government takes away some of the consumption so the overall effect on welfare for the old generation at t = 50 is uncertain. Apart from this, figure 4.6 give same picture as in figure 3.5 from chapter 3, when we move the economy from a low level steady state to high level steady state. The fast learners are better off during the entire learning transition. The results from proposition 3.3 in chapter 3 might be used here, due to the increasing returns in the production function which shifts the welfare upwards and the decreasing speed of learning tend to smooth the "curves" in figure 4.6 when t is large, but this has to be investigated properly.

We can compare this situation to the case, where the two groups of agents have a constant speed of learning. In this case the agents use a constant speed of learning, given by $a_f$ and $a_s$ respectively. The welfare comparison between fast and slow agents in this situation looks as follows, given the parameters from footnote 5 and set spending $\gamma$ equal to 0.5 for all t.

Figure 4.7 about here.[6]

As can be seen from figure 4.7, we cannot distinguish between the two groups when the economy is close to the steady state, since both groups produce a noisy forecast such that the welfare fluctuates around the steady state level, hence the welfare results from chapter 3 does not hold in this case.

We are left with two different results, if we use either decreasing speeds of learning, or constant speeds of learning. The reason for using the decreasing speed of learning was to be sure that the steady state was locally stable under learning. However, when we use decreasing speeds both groups of agents become very slow when t is large, and the difference between fast and slow learners might become negligible when t is large. The other approach is to use constant speeds as in chapter 2 and 3, but the economy might fluctuate between a low level steady state and a high level steady state even if there is no changes in policy and no structural changes, as shown in figure 4.4. However, the response from the agents to a change, for example, in government spending is much more rapid than with a constant speed of learning, than a decreasing speed of learning, but the welfare comparison is not valid when t is large.

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[6] $\gamma = 0.5$ and $a_f = 0.8$, $a_s = 0.3$.

## 4.7. Conclusion.

In this chapter we have studied the overlapping generations model from the previous chapters with intrinsic noise added to the model. This made the model somewhat more complicated, stability issues became more difficult to investigate and we could study only local stability of the steady state under learning. In order to obtain local stability it was necessary to a decreasing speed of learning. It was possible to generate fluctuations without policy changes or other structural changes if the agents used a constant speed of learning.

The welfare comparison between fast and slow learners are not similar to the results in chapter 2, when the agents use a decreasing speed of learning as shown by figure 4.2b. If the agents use a constant speed of learning, it was not possible to distinguish between fast and slow. This is due to the agents producing noisy forecast even in the limit. Hence we have to choose whether to use a constant speed of learning or a decreasing speed of learning. A reason for choosing the constant gain parameter was that it might an equilibrium learning rule, in the sense that it minimised the mean square forecast error, when all agents use the same speed of learning $\boldsymbol{a}$. We could try to establish the existence of an equilibrium learning rule, and find other criteria for the equilibrium learning rule, based on the welfare function for example. We could also allow for a variance in $\boldsymbol{a}$, but this is left for further investigation.

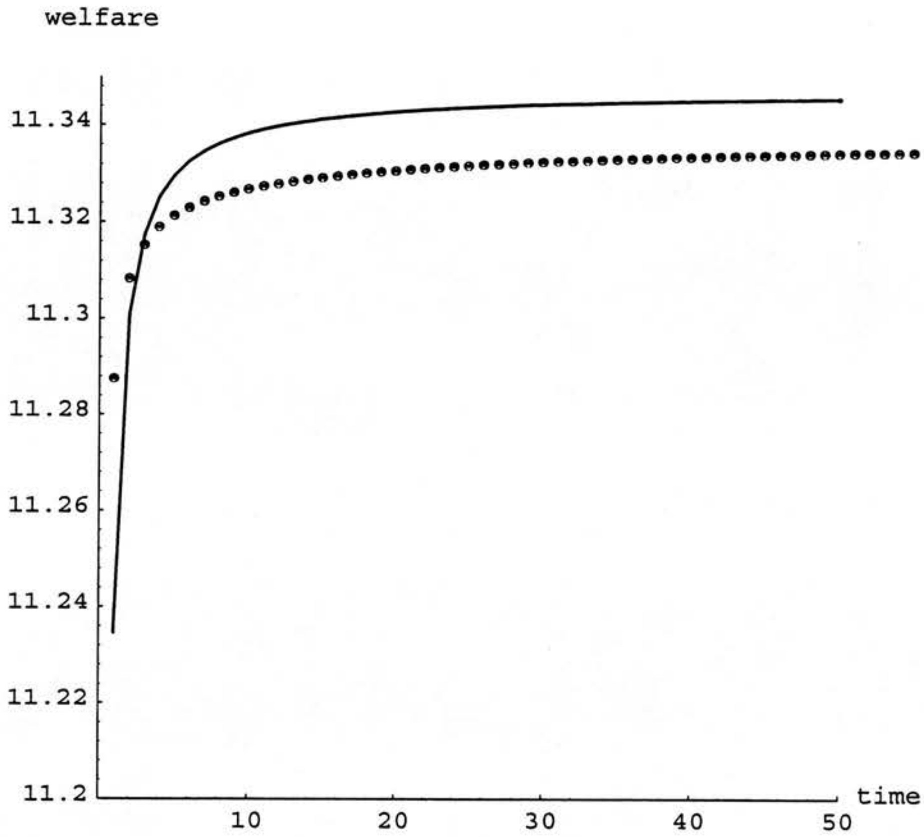



**Figure 4.1 The welfare paths for the slow learners (dotted line) and the fast learners (straight line).**

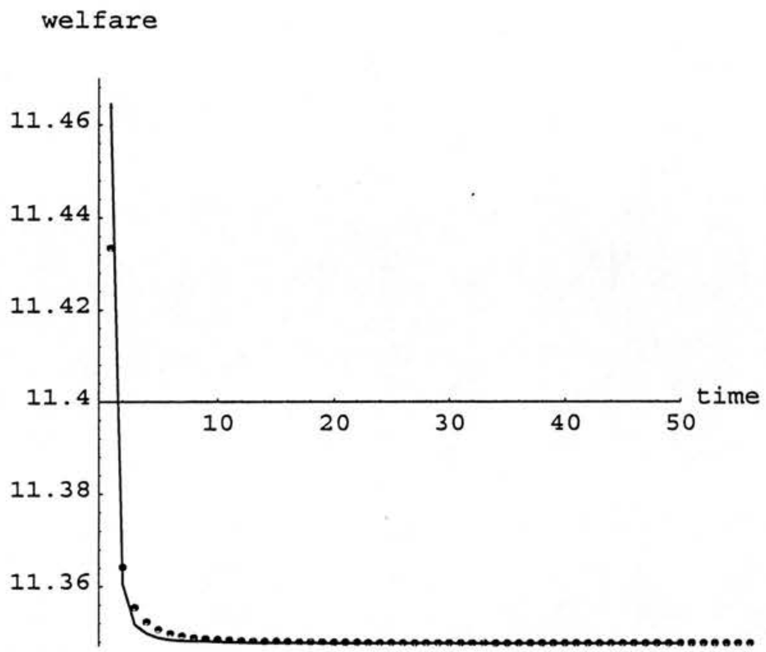



**Figure 4.2a. The welfare paths for the slow learners (dotted line) and the fast learners (straight line).**

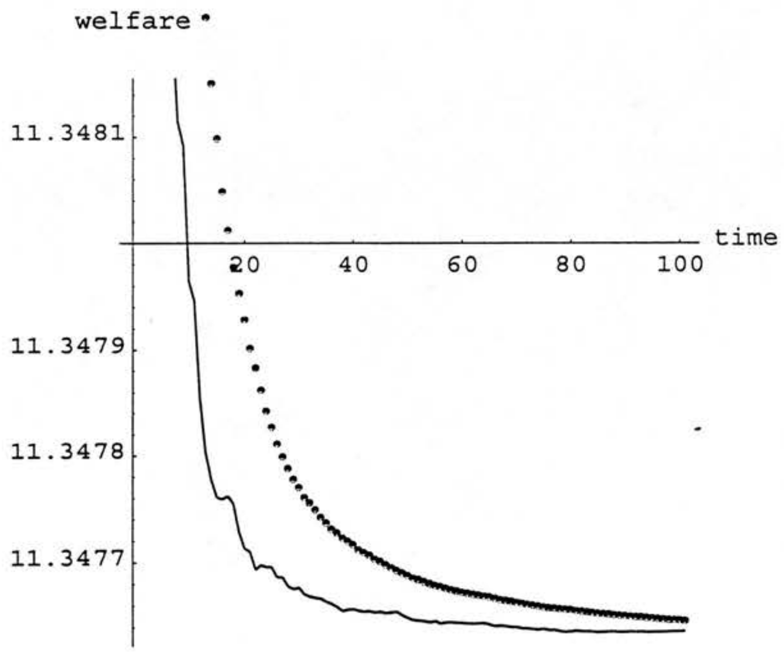



Figure 4.2a with a different scaling on y-axis.

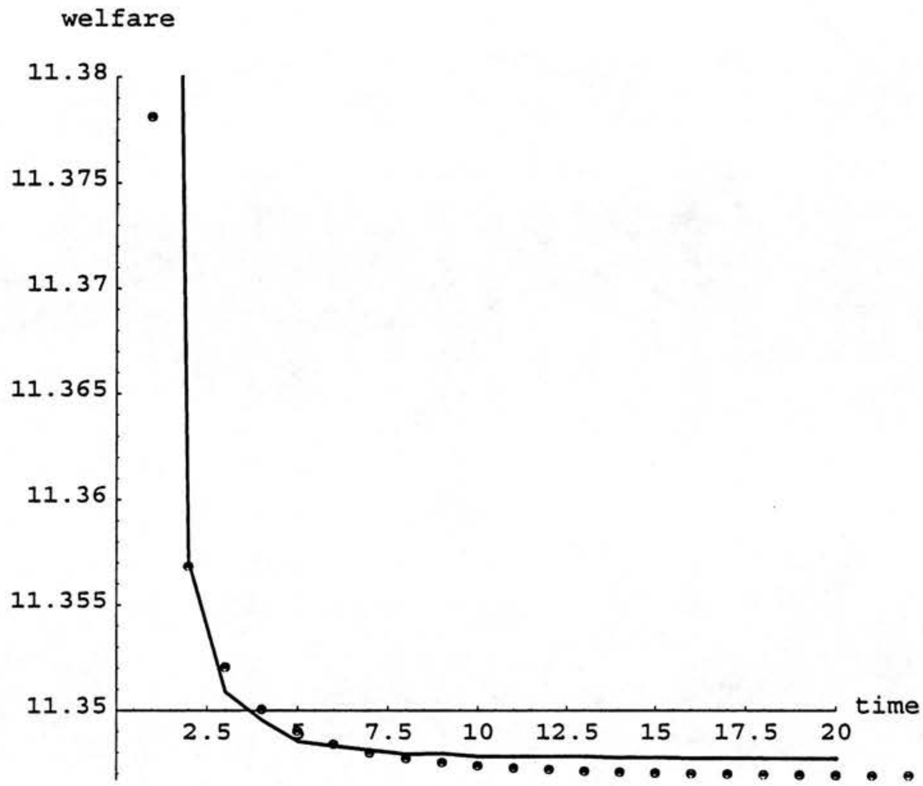



**Figure 4.2b. The welfare paths for the slow learners (dotted line) and the fast learners (straight line)**

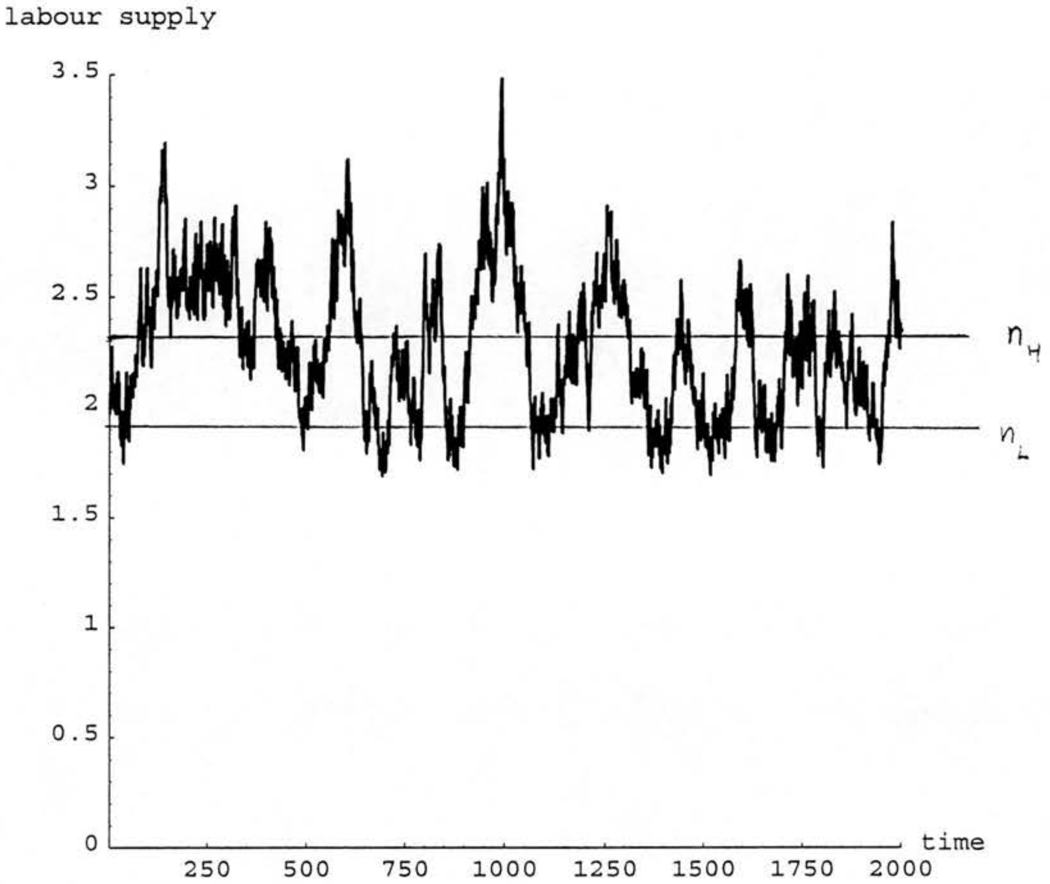



Figure 4.4. The labour supply, when $n_L = 1.9$, $n_H = 2.31$, and $a = 0.8$.No govt. intervention.

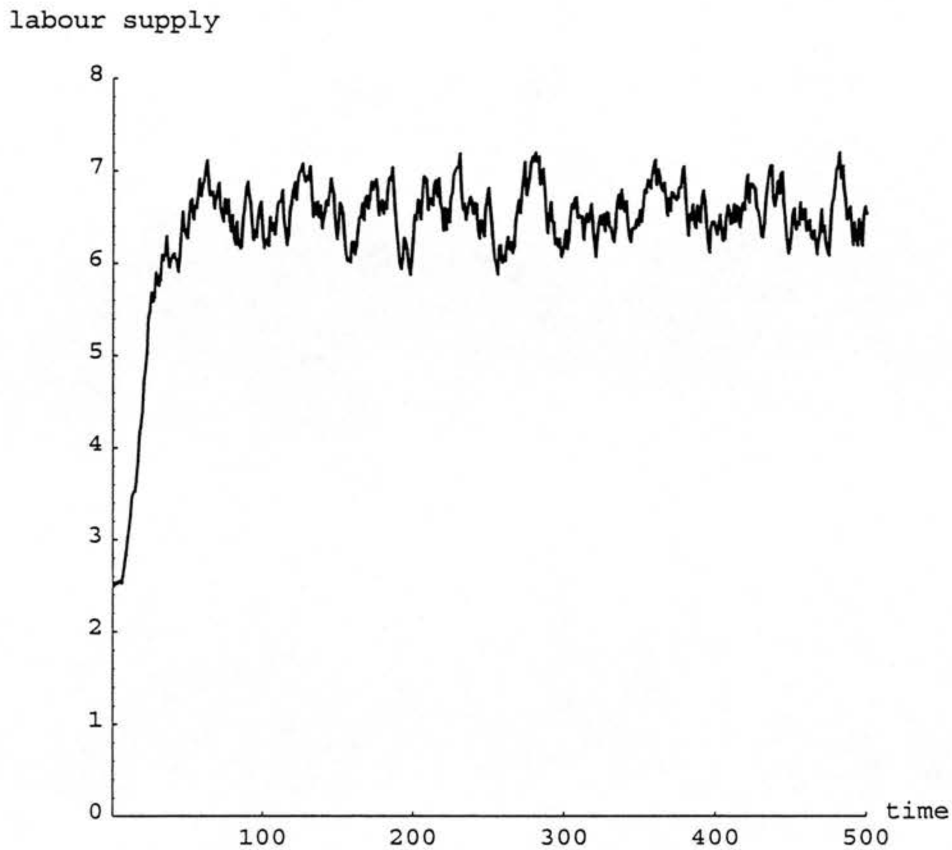



**Figure 4.5. The labour supply when there are multiple steady states and a constant speed of learning.**

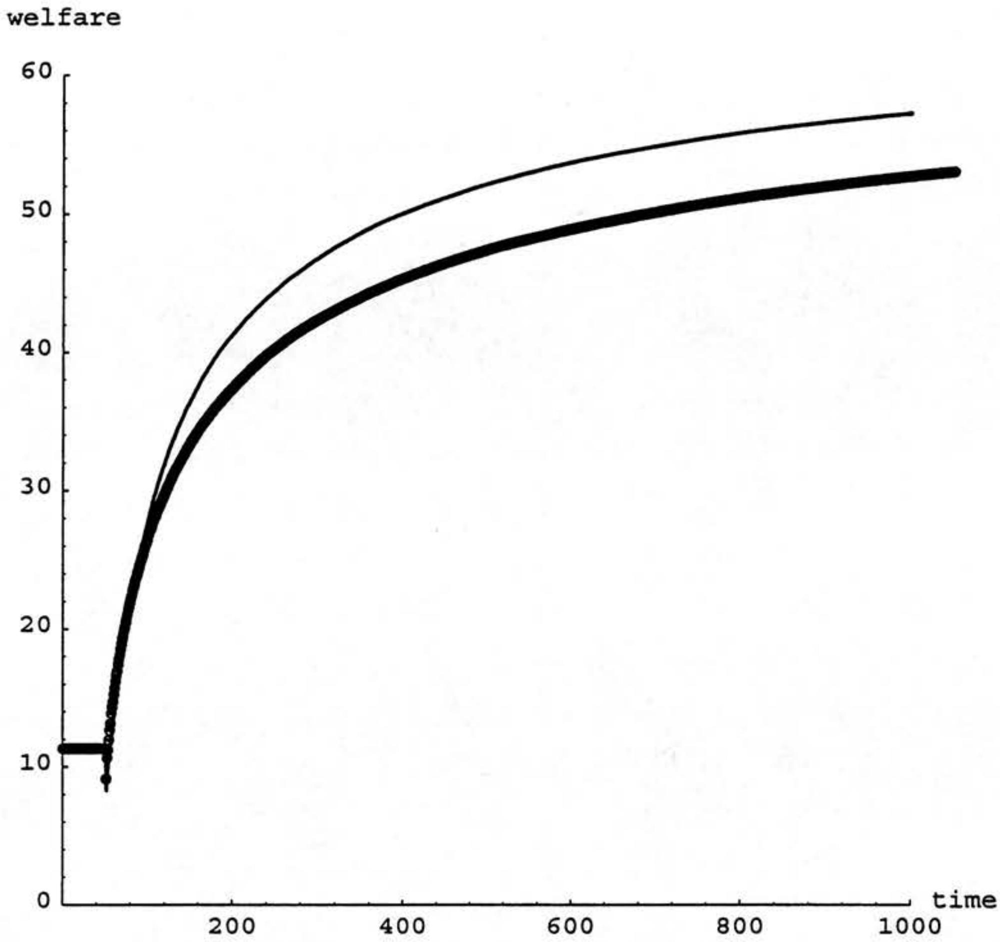



**Figure 4.6. The welfare paths for the slow learners (thick line) and the fast learners (thin line).**

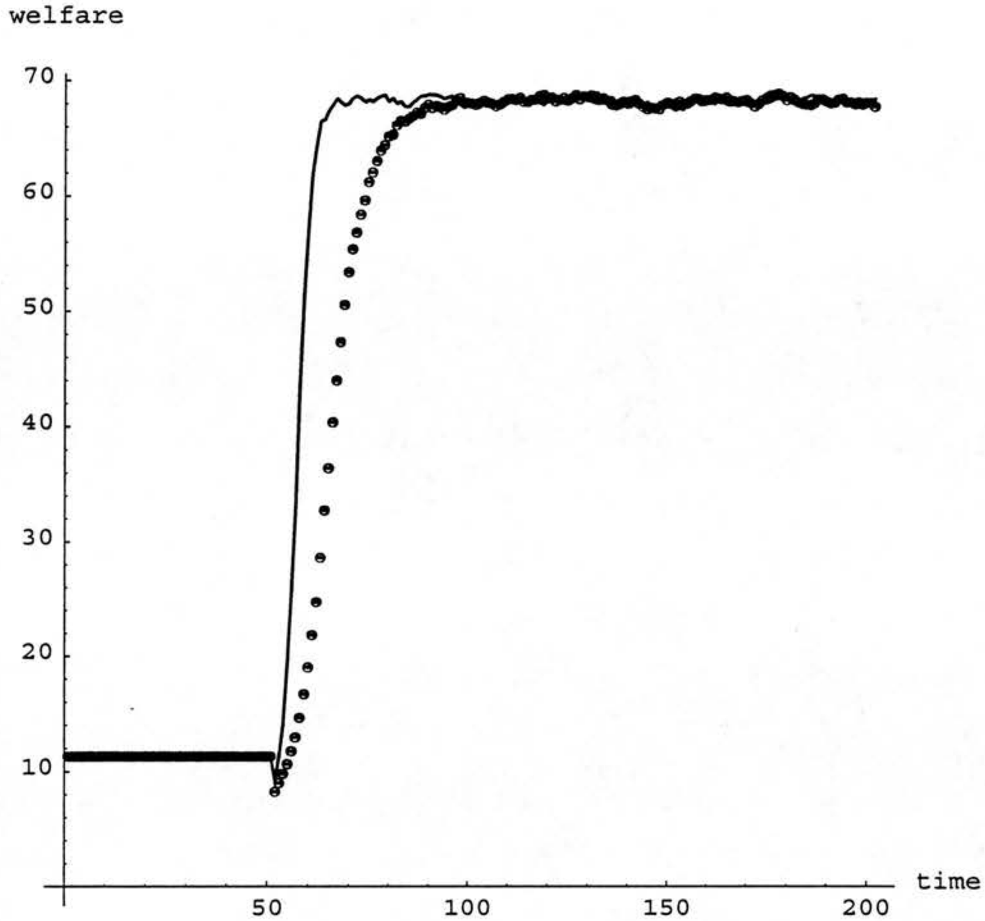



**Figure 4.7. The welfare paths for the slow learners (dotted line) and the fast learners (straight line).**

## 5.1. Introduction.

In macroeconomic models where the economic variables depend on agents forecast of future variables, it is usually assumed that agents have rational expectations. However, if the agents do not have rational expectations initially, how can they obtain rational expectations. One way to solve this problem has been to introduce an adaptive learning rule in the model, and study the stability of the rational expectations equilibrium under learning, see for example Bray (1982), Lucas (1986), Guesnerie and Woodford (1991), and Evans and Honkapohja (1995b). It has been standard to assume that all agents have the same learning rule such that there are homogeneous agents. There are a few papers where heterogeneous agents are incorporated in the model, Bray and Savin (1986) studies the cobweb model where the agents have different initial priors but all agents have the same learning rule, Marcet and Sargent (1989) analyse recursive least squares learning in model with hidden state variables, and use the model from Townsend (1983) as an example. Evans, Honkapohja and Marimon (1995) look at heterogeneous learning rules and experimentation in an overlapping generations model with a continuum of agents. Frydman (1982) and several papers in Frydman and Phelps (1983) also considers the problem with heterogeneous agents, where the agents have to know the average opinion of prices. In the model presented here an agent does not need to know the average opinion of expected prices. The agent only needs information about the temporary equilibrium price.

We analyze a standard version of the overlapping generations model that have more than one type of agents, for convenience we initially study a model with two types of agents, but this can be easily extended to $m$ types of agents. At time $t$ the agents are forecasting the price at $t + 1$ according to an adaptive learning rule, we only study adaptive learning or "irrational" learning in this paper. The difference between the agents is in the speed of learning. One type of agents have a high speed of learning. These are called fast learners and a fast learner are born into families of fast learners, we assume there is a fraction $\mu$ fast learners at time $t$ where $\mu \in [0,1]$. The other type agents have a low speed of learning and are called slow learners, and a slow learner are born into families of slow learners. There is a fraction of $1 - \mu$ slow learners at time $t$. The agents cannot change their type during the learning transition.

First, we study the stability of a stationary rational expectations equilibrium under heterogeneous learning. There is no uncertainty in this model and the stationary rational expectations equilibrium is the steady state. The conditions for local stability of rational expectations equilibrium under learning depend on the fraction of learners. Furthermore, there is a correspondence between the local stability conditions in the homogenous case and the stability conditions in the heterogeneous case. We should note that in the limit the two types of agents are identical, since they are at the steady state. Given an extra assumption on the equilibrium mapping, we can show global stability of the steady state under learning.

Second, we compare the welfare between a fast agent and a slow agent born at time $t$ during the learning transition to the rational expectations equilibrium. It is shown that the fast learners are better off than the slow learners for $t$ sufficiently large. The intuition behind this result is that the fast learners are making a "better" forecast than the slow learners for $t$ sufficiently large. Since the fast learners have a higher speed of learning they have a higher speed of convergence, and when $t$ is sufficiently large they are closer to the actual price, because the actual price depends on the forecast of the agents. Thus the fast agents are "closer" to solving the true optimisation problem when $t$ is sufficiently large. However, it is possible that the slow agents are better off during the early periods of the learning transition, but eventually the fast agents overtake the slow agents and become better off, this is illustrated by numeric examples. In this set-up the fast agent is ignorant of the slow agent, and does not take into account that the other agent is using a learning rule as well.

We also the effect of a one-time increase in the money supply. The result is not surprising, the rational expectations equilibrium price increases and the agents update their forecast accordingly. We can use the welfare analysis from above to compare welfare between types. An extension of the model to $m$ different types of learners is discussed in the last section of the paper. The results from the 2-type case are still valid with the natural extension of assumptions. The chapter is organised as follows, section 5.2 outline the model and in section 5.3 we look at the local and global stability of a steady state. In Section 5.4 the welfare comparison is made. The effect of a change in money supply is analysed in section 5.5, and in section 5.6 we extend the model to $m$ different types of learners. Section 5.7 contains the conclusions.

## 5.2. The model.

The model is a version of the standard overlapping generations model, where the agents live for two periods and there is one good there is produced and consumed. An agent work at time $t$ and consume at time $t + 1$. The production function $q$ is assumed to be linear $q(n) = n$, and one unit of output $y$ yields one unit of the consumption good $c$, $y = c$. We can then use the same notation $p_t$ for wages and prices. There is a constant supply of money, $M > 0$. The agent born at time $t$ demands money at time $t$ and use the money to buy consumption at time $t + 1$. The welfare function $W$ is assumed to be separable in labour supply at time $t$, $n_t$, and consumption at time $t + 1$, $c_{t+1}$, $W = U(c_{t+1}) - V(n_t)$, where $U$ is the utility function and $V$ the labour function. $U$ is a strictly concave and increasing $C^2$-function for all $c > 0$ and $V$ is a strictly convex and increasing $C^2$-function for all $n > 0$. There is no uncertainty and the agents have point expectations about future price.

There are $m$ different agents, and we that a fraction of $\mu_i$ agents born at time $t$ are of type $i$ where $\mu_i \in [0,1]$ for all $i$ and $\sum_{i=1}^{m} \mu_i = 1$. Type $i$-agents are born into families of type $i$-agents and the $i$-th agent solve the following maximisation problem

$$\max_{c_{t+1}^i, n_t^i, M_t^i} \; U(c_{t+1}^i) - V(n_t^i)$$

$$\text{subject to :} \tag{5.1}$$

$$p_t n_t^i = M_t^i \text{ and } p_{t+1}^{e,i} c_{t+1}^i = M_t^i .$$

where $M_t^i$ is the money demand at time $t$, $p_t$ is the price at time $t$ and $p_{t+1}^{e,i}$ is the $i$-th agent's forecast of the price at time $t + 1$. This forecast is made at time $t$. The first-order condition is given by

$$U'(\frac{p_t}{p_{t+1}^{e,i}} n_t^i)\frac{p_t}{p_{t+1}^{e,i}} = V'(n_t^i) \tag{5.2}$$

for $i = 1,...,m$. Given the assumptions on $U$ and $V$ the second-order condition for a maximum is satisfied. (5.2) is rearranged such that the labour supply $n_t^i$ is a function of $p_t/p_{t+1}^{e,i}$ the expected real interest rate, this also represents the expected real wage

$$n_t^i = \xi\left(\frac{p_t}{p_{t+1}^{e,i}}\right) \tag{5.3}$$

for $i = 1,...,m$, where $\xi$ is a $C^1$-function of $p_t/p_{t+1}^{e,i}$ according to the assumptions on $U$ and $V$.

**Assumption 5.1.** $\xi$ is an increasing function in $p_t/p_{t+1}^{e,i}$ for $i = 1,....,m$.

According to assumption 5.1, an increase in the expected real interest rate (or expected real wage) $p_t/p_{t+1}^{e,i}$ gives an increase in the labour supply $n_t^i$. In this case the substitution-effect dominates the income-effect, because an increase in the expected real wage cause the agents to substitute leisure for labour, thus increasing labour supply. If $xU'(x)$ is increasing in $x$, then $\xi$ satisfies assumption 5.1, because an increase in $p_t/p_{t+1}^{e,i}$ implies an increase $(p_t/p_{t+1}^{e,i})\,U'((p_t/\,p_{t+1}^{e,i})\,n_t^i)$, the left-hand side of the first order condition (2). Since $V'' > 0$ the right hand-side must increase as well and the labour supply must increase as well. The condition that $xU'(x)$ is increasing can be expressed as

$$\lambda(x) = -\frac{xU''(x)}{U'(x)} < 1.$$

$\lambda(x)$ is the well-known Arrow-Pratt measure of relative risk aversion. If $\lambda(x) < 1$ then cycles of period $k \geq 2$ cannot exist as shown in Grandmont (1985).

The money supply $M$ is constant for all $t$, and we have the following definition of a temporary equilibrium.

**Definition 5.1.** A temporary equilibrium is a $(m + 1)$-tuple $(p_t, \{n_t^i\}_{i=1}^m)$ such that $n_t^i$ is a solution to (5.1) for all $t$ and the money supply is equal to the money demand

$$M = \sum_{i=1}^m \mu_i M_t^i \qquad \text{for all t.} \tag{5.4}$$

If we combine (5.4) with the budget constraints at time $t$, we have:

$$\frac{M}{p_t} = \sum_{i=1}^m \mu_i n_t^i \tag{5.5}$$

Insert (5.3) into (5.5), we can express the actual price $p_t$ as a function of the expected prices :

$$\frac{M}{p_t} = \sum_{i=1}^m \mu_i \xi\left(\frac{p_t}{p_{t+1}^{e,i}}\right) \tag{5.6}$$

Since $\xi$ is a $C^1$-function with $\xi' > 0$, the implicit function theorem can be used to express $p_t$ as a function of $p_{t+1}^{e,i}$, $i = 1,....m$ :

$$p_t = H(p_{t+1}^{e,1},.., p_{t+1}^{e,m}) \quad \text{for all } t. \tag{5.7}$$

and $H$ is $C^1$-function in $p_{t+1}^{e,i}$, $i = 1,...,m$, according to the implicit function theorem. However the implicit function theorem only defines the function $H$ locally, thus we make the following regularity assumption on $H$.

**Assumption 5.2.** $H$ is a well-defined $C^1$-function for all $p_{t+1}^{e,i} > 0$ and $i = 1,...,m$.

Given this assumption and assumption 5.1 on the function $\xi$, we can study the monotonicity of $H$.

**Lemma 1.** *Given assumptions* 5.1 *and* 5.2, *then the partial derivative of* $H$ *w.r.t.* $p_{t+1}^{e,i}$ *is positive*

$$H_i(p_{t+1}^{e,1},.., p_{t+1}^{e,m}) > 0 \qquad \textit{for } i = 1,.....,m.$$

**Proof.** See appendix. ■

The function $H$ is increasing in the expected prices such that an increase in $p_{t+1}^{e,i}$,leads to an increase in $p_t$. For example, an increase in $p_{t+1}^{e,i}$ results in a decrease of the expected real wage, $p_t \,/\, p_{t+1}^{e,i}$. The substitution effect dominates the income effect agent $i$'s labour supply, $n_t^i$, decrease and there is a decrease in aggregate labour supply $n_t$. This leads to an increase in $p_t$, because $n_t = M/p_t$. A perfect foresight equilibrium is a sequence $\{p_t\}_t$ that satisfies the following condition :

$$p_t = H(p_{t+1},...,p_{t+1}) \qquad \text{for all } t.$$

A steady state is a $\bar{p}$ where $\bar{p} = H(\bar{p},..., \bar{p})$. Let us briefly study the case where all agents have perfect foresight, since this gives a condition for the existence of an interior steady state. If all agents have perfect foresight $p_{t+1}^{e,i} = p_{t+1}$ for all $i$, then the agents are identical. (5.4) is reduced to $M = M_t$ and if we combine $p_t n_t = M$ with the first order condition (5.2) we have

$$U'(\frac{M}{p_{t+1}})\frac{1}{p_{t+1}} = \frac{1}{p_t}V'(\frac{M}{p_t}) \tag{5.8}$$

Since the right-hand side of (5.8) is monotone in $p_t$, we can write $p_t$ is a function of $p_{t+1}$:

$$p_t = F(p_{t+1}) \tag{5.9}$$

The function $H$ is reduced to the function $F$ when agents have perfect foresight. $F$ is $C^1$-function for all $p_{t+1} > 0$. The perfect foresight equilibrium $\{p_t\}_t$ satisfies (5.9). The steady state $\bar{p}$ satisfies $\bar{p} = F(\bar{p})$, i.e. an $\bar{n}$ where $U'(\bar{n}) = V'(\bar{n})$, since all agents have the same welfare function. If we furthermore assume that $U'(c) \to \infty$ when $c \to 0$ and $V'(n) \to \infty$ when $n \to \infty$ then there exists a unique interior steady state $\bar{p} > 0$. If $cU'(c) \to 0$ when $c \to 0$, then the autarky steady state $\bar{p} = 0$ also exists. There are a continuum of perfect foresight paths in this economy and the steady state $\bar{p} > 0$ is unstable under perfect foresight, let $\{p_t\}_t$ be a perfect foresight path with $p_0 > \bar{p}$, then $p_t$ goes to infinity for $t$ going to infinity. If $p_0 < \bar{p}$, then $p_t$ goes to 0 as $t \to \infty$. In the next section we study learning and the stability of $\bar{p}$ under learning.

## 5.3. The conditions for local and global stability of the steady state under learning.

The learning behaviour we introduce is adaptive learning. In formulating a forecast of the price at time $t + 1$, $p_{t+1}^{e,i}$, the $i$-th agent is using the forecast at time $t - 1$, $p_t^{e,i}$, the actual price at time $t - 1$, $p_{t-1}$. We use $p_{t-1}$ instead of $p_t$ in order to avoid simultaneity between $p_{t+1}^{e,i}$ and $p_t$ as explained below. There is a choice whether the agents should learn about the real interest rate or the price. One reason for learning on the real interest rate is that we can allow for the government to change the money supply over time. If the money supply increases, the price increases as shown in section 5 below, and the expected prices just increase over time. We have to look at the real interest rate instead to get convergence to a steady state. However, to keep things simple the agents learn about price and for convenience we set $m = 2$. The difference between the two types is in the speed of learning. A fraction of $\mu$ agents born at time $t$ use a high speed learning rule into and a fraction of $1 - \mu$ agents use a low speed learning rule, where $\mu \in [0,1]$. The high speed agents update $p_{t+1}$ according to the learning rule

$$p_{t+1}^{e,f} = p_t^{e,f} + a_t^f \left(p_{t-1} - p_t^{e,f}\right) \tag{5.10}$$

where $p_{t+1}^{e,f}$ is the forecast made at time $t$ of the price at time $t + 1$. The slow speed agents update $p_{t+1}$ according to the learning rule

$$p_{t+1}^{e,s} = p_t^{e,s} + a_t^s \left(p_{t-1} - p_t^{e,s}\right) \tag{5.11}$$

where $p_{t+1}^{e,s}$ is the forecast made at time $t$ of the price at time $t + 1$. The difference between the two types of agents are in the speed of learning $a_t^i$, $i = s,f$ and we assume

$$0 < a_t^s < a_t^f \leq 1 \qquad \text{for all } t.$$

$$\sum_{t=1}^{\infty} a_t^i = \infty \qquad \text{for } i = s,f.$$

Since there is no uncertainty in the model it is not necessary to assume that $a_t^i \to 0$ when $t$ goes to infinity. The high speed agents are denoted by "$f$" and are called *fast learners*, while the slow speed agents are denoted by "$s$" and are called *slow learners*. The terms slow and fast merely describes the difference in the speed of learning, the fast learners do not become perfect foresight agents suddenly as a result of a higher speed of learning. There has been a wide interest in the literature in the learning rules (10) and (11), see e.g. Evans and Honkapohja (1995b) for a recent overview of adaptive learning rules in dynamic macroeconomic models. The learning rules (10) and (11) are simple although they are ad hoc. If the steady state is stable under learning, the rate of convergence is fast if the speed of learning is high, thus the agents are only misspecified for a short number of periods. Hence the learning rule is "reasonable" as noted by Bray (1982). In the following we use constant speeds of learning, $a_t^i = a_i$, $i = s,f$ and $a_s < a_f$. A constant speed of learning is sometimes called a constant gain parameter, and we place a high weight on current observations. A decreasing speed of learning place a low weight on current observations when $t$ is large, while earlier observations have a higher weight. The choice of a constant speed of learning does not change the results below and is used to simplify the notation. However if there is uncertainty in the model, for example a technology shock or a preference shock, then a constant speed of learning produces a noisy forecast even in the limit. In this case the economy can fluctuate around the steady state depending on the size of shock. We assume that $\mu$ is exogenous, an agent is either born as a slow or a fast learner and cannot change the speed of learning. A possible extension could be to make $\mu$ endogenous, for example it could depend on the welfare the agents receive. If the welfare of slow learner compared to the welfare of a fast learner is decreasing the proportion of slow learners would decrease, and we can study evolutionary aspects in this model.

If we insert the temporary equilibrium condition $p_t = H(p_{t+1}^{e,f}, p_{t+1}^{e,s})$ into the learning rules (5.10) and (5.11), we have the non-linear dynamic system :

$$
\begin{aligned}
p_{t+1}^{e} &= p_t^{e,f} + a_f( H(p_t^{e,f}, p_t^{e,s}) - p_t^{e,f} ) \\
p_{t+1}^{e,s} &= p_t^{e,s} + a_s( H(p_t^{e,f}, p_t^{e,s}) - p_t^{e,s} )
\end{aligned}
\tag{5.12}
$$

If the actual price $p_t$ was used instead of $p_{t-1}$ then we would have to determine $p_t$ and $p_{t+1}^{e,i}$ simultaneously, thus for convenience we use $p_{t-1}$ . We could use $p_t$ if we are only concerned by local stability see Evans and Honkapohja (1995b), but for the global stability and welfare comparison we need the recursive system (5.12). When the agents are using the learning rule, agents do not take into account that the other agent is making an error in his forecast. An agent does not need to know the function $H$ or the forecast from the other agent. The agent just needs information of the actual price $p_t$ and this assumed to be announced to both agents. It might seem strange that the agents do not take into account that other agents are using a misspecified model, and are ignorant of this element, but I have not seen papers where agents a learning rules like (5.10) and (5.11), and using this information in their forecast. There is a lot bounded rationality here. We could assume that one type of agent had perfect foresight and the other used the learning rule, but that would make the arithmetic's much harder, see Evans, Honkapohja and Sargent (1993). The dynamic system (5.12) can be written as follows

$$
(p_{t+1}^{e,f}, p_{t+1}^{e,s}) = G(p_t^{e,f}, p_t^{e,s}) \tag{5.13}
$$

where $G : R_+^2 \rightarrow R_+^2$ is a $C^1$-function in $(p_t^{e,f}, p_t^{e,s})$ and given by (5.12). The conditions for local stability of the steady state $(\overline{p}, \overline{p})$ under learning are determined by the size of the eigenvalues for the Jacobian matrix of the function $G$ at $(\overline{p}, \overline{p})$ :

$$
J(\overline{p}, \overline{p}) = \begin{pmatrix} 1 + a_f(H_1(\overline{p},\overline{p}) - 1) & a_f H_2(\overline{p},\overline{p}) \\ a_s H_1(\overline{p},\overline{p}) & 1 + a_s(H_2(\overline{p},\overline{p}) - 1) \end{pmatrix}.
$$

If the eigenvalues of $J$ are less than 1 in absolute value, then $(\overline{p}, \overline{p})$ is locally stable under learning (5.10) and (5.11). The characteristic polynomial $\pi(\lambda)$ for $J(\overline{p}, \overline{p})$ is given by:

$$
\pi(\lambda) = \lambda^2 - (Tr\, J)\lambda + (Det\, J)
$$

where $Tr\,J$ is the trace of $J$ and $Det\,J$ is the determinant of $J$

$$Tr\,J = 1 + a_f(H_1(\bar{p},\bar{p}) - 1) + 1 + a_s(H_2(\bar{p},\bar{p}) - 1)$$

$$= (1 - a_f) + (1 - a_s) + a_f H_1(\bar{p},\bar{p}) + a_s H_2(\bar{p},\bar{p})$$

$$Det\,J = (1 + a_f(H_1(\bar{p},\bar{p}) - 1))(1 + a_s(H_2(\bar{p},\bar{p}) - 1)) - a_f H_1(\bar{p},\bar{p})\, a_s H_2(\bar{p},\bar{p})$$

$$= (1 - a_f)(1 - a_s) + (1 - a_f) a_s H_2(\bar{p},\bar{p}) + (1 - a_s) a_f H_1(\bar{p},\bar{p},)$$

It is easy to see that $Tr\,J > 0$ and $Det\,J > 0$, when the partial derivatives $H_1$ and $H_2$ are positive. Since the eigenvalues of $J$ are the roots of the characteristic polynomial, we have to find the roots of $\pi$. The roots of $\pi$ are as follows:

$$\lambda_1 = \frac{(Tr\,J) + \sqrt{(Tr\,J)^2 - 4(Det\,J)}}{2} \quad \text{and} \quad \lambda_2 = \frac{(Tr\,J) - \sqrt{(Tr\,J)^2 - 4(Det\,J)}}{2}$$

$\lambda_1$ and $\lambda_2$ are real if $(Tr\,J)^2 - 4(Det\,J) > 0$. This can be shown by tedious calculations and the roots are real. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then the steady state is locally stable. It is easy to see that $\lambda_2 < \lambda_1$, hence we have to show:

$$-1 < \lambda_2 < \lambda_1 < 1.$$

This is satisfied if

$$0 < 1 - H_1(\bar{p},\bar{p}) - H_2(\bar{p},\bar{p}) \tag{5.14}$$

as shown in the appendix. As a result of the symmetry in the function $H$ at $\bar{p}$ we have

$$H_2(\bar{p},\bar{p}) = ((1 - \mu)/\mu)\, H_1(\bar{p},\bar{p}),$$

this shown in the appendix. Then (5.14) is reduced to

$$H_1(\bar{p},\bar{p}) < \mu$$

We can summarise the result in proposition 1.

**Proposition 5.1.** *Given assumptions 5.1 and 5.2. If* $H_1(\bar{p},\bar{p}) < \mu$ *then the steady state* $\bar{p}$ *is locally stable under learning.*

In the homogeneous case, where either $\mu = 1$ or $\mu = 0$, the condition for local stability are either $H_1(\bar{p},\bar{p}) < 1$ (or $H_2(\bar{p},\bar{p}) < 1$). We can show that the steady state is locally stable in the homogeneous case if and only if it is locally stable in the heterogeneous case $0 < \mu < 1$. Let us assume that $\mu = 1$, then temporary equilibrium is reduced to $p_t = F(p_{t+1}^{e,f})$. The steady state $\overline{p}$ is locally stable under learning if $F'(\overline{p}) < 1$. In order to obtain an equivalence between stability in the homogeneous case and heterogeneous case, look at the derivative of $p_t$ w.r.t $p_{t+1}^{e}$ when $\mu = 1$:

$$\frac{\partial(M / p_t)}{\partial p_{t+1}^{e,f}} = \xi'(p_t / p_{t+1}^{e,f}) \frac{p_{t+1}^{e,f}(\partial p_t / \partial p_{t+1}^{e,f}) - p_t}{(p_{t+1}^{e,f})^2} \quad \Rightarrow$$

$$\left.\frac{\partial p_t}{\partial p_{t+1}^{e,f}}\right|_{\mu=1} = \frac{1}{\Delta} \xi'\left(\frac{p_t}{p_{t+1}^{e,f}}\right)\left(\frac{p_t}{p_{t+1}^{e,f}}\right)^2 \tag{5.15.a}$$

and $\Delta = \left( \frac{M}{p_t} + \xi'\left(\frac{p_t}{p_{t+1}^{e,f}}\right)\frac{p_t}{p_{t+1}^{e,f}} \right)$. If we look at the partial derivative of $p_t$ w.r.t. $p_{t+1}^{e,f}$, when $0 < \mu < 1$ we have from the proof of lemma 1, that:

$$\left.\frac{\partial p_t}{\partial p_{t+1}^{e,f}}\right|_{0<\mu<1} = \frac{1}{\Lambda} \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,f}}\right)\left(\frac{p_t}{p_{t+1}^{e,f}}\right)^2 \tag{5.16.a}$$

where $\Lambda = \left( \frac{M}{p_t} + \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,f}}\right)\frac{p_t}{p_{t+1}^{e,f}} + (1-\mu)\xi'\left(\frac{p_t}{p_{t+1}^{e,s}}\right)\frac{p_t}{p_{t+1}^{e,s}} \right)$. At $\overline{p}$, (5.15a) is equal to :

$$\left.\frac{\partial p_t}{\partial p_{t+1}^{e,f}}(\overline{p})\right|_{\mu=1} = \frac{1}{\Delta} \xi'(1). \tag{5.15.b}$$

where $\Delta = M / \overline{p} + \xi'(1)$ and (5.16a) is equal to

$$\left.\frac{\partial p_t}{\partial p_{t+1}^{e,f}}(\overline{p})\right|_{0<\mu<1} = \frac{1}{\Lambda} \mu\xi'(1) = \mu(\frac{1}{\Lambda} \xi'(1)) \tag{5.16.b}$$

where $\Lambda = M / \overline{p} + \xi'(1)$. Hence (5.16b) can be written as function of (5.15b), and we have the following connection between $H_1$ and $F'$ at the steady state $\overline{p}$ :

$$H_1(\overline{p}, \overline{p}) = \mu F'(\overline{p})$$

Hence, if the steady state is locally stable in the homogeneous case, $F'(\overline{p}) < 1$, then it is also stable in the heterogeneous case, $H_1(\overline{p}, \overline{p}) < \mu$. The opposite result also holds, if $H_1(\overline{p}, \overline{p}) < \mu$ then $\mu F'(\overline{p}) < \mu$ and $F'(\overline{p}) < 1$. Thus local stability in the heterogeneous case implies local stability in the homogeneous case. It is not surprising that there is an equivalence between the economy with homogeneous agents and heterogeneous agents in this set-up, because if the economy is in a steady state both types of agents makes the same forecast and we cannot distinguish between the two types.

In order to illustrate the dynamics and simplify the analysis, let us assume that the utility function $U$, the labour function $V$ have the following functional form :

$$U(c) = \frac{1}{1-\sigma} c^{1-\sigma}$$

$$V(n) = \frac{1}{1-\varepsilon} n^{1-\varepsilon}$$

where $0 < \sigma < 1$ and $\varepsilon > 0$. Given these functions, we can derive an explicit expression for the function $H(p_{t+1}^{e,f}, p_{t+1}^{e,s})$ :

$$H(p_{t+1}^{e,f}, p_{t+1}^{e,s}) = \left( \frac{1}{M} \left[ \mu (p_{t+1}^{e,f})^{-\frac{1-\sigma}{\varepsilon+\sigma}} + (1-\mu)(p_{t+1}^{e,s})^{-\frac{1-\sigma}{\varepsilon+\sigma}} \right] \right)^{-\frac{\varepsilon+\sigma}{1+\varepsilon}} \tag{5.17}$$

In the case where $p_{t+1}^{e,f} = p_{t+1}^{e,s}$ $H$ is reduced to

$$F(p_{t+1}^{e,f}) = M^{\frac{\varepsilon+\sigma}{1+\varepsilon}} (p_{t+1}^{e,f})^{\frac{1-\sigma}{1+\varepsilon}} \tag{5.18}$$

If the agents have perfect foresight the equilibrium condition is given by (5.18) with $p_{t+1}^{e,f} = p_{t+1}^{e,s} = p_{t+1}$ otherwise it is given by (5.17) as long as $p_{t+1}^{e,f} \neq p_{t+1}^{e,s}$. It is easy to show that the partial derivatives are positive $H_1(p_{t+1}^{e,f}, p_{t+1}^{e,s}) > 0$ and $H_2(p_{t+1}^{e,f}, p_{t+1}^{e,s}) > 0$ and

$$H_2(\bar{p}, \bar{p}) = (1 - \mu/\mu) H_1(\bar{p}, \bar{p})$$

Let $H_{ij} = \partial^2 H(p^{i,e}, p^{j,e}) / \partial p^{i,e} \partial p^{j,e}$ denote the second-order partial derivatives of the function $H$ for $i,j = 1,2$. It can be shown that $H_{11} < 0$, $H_{22} < 0$, $H_{12} > 0$, $H_{21} > 0$ and $H_{11} H_{22} - H_{12} H_{21} > 0$ such that $H$ is strictly concave in $(p_{t+1}^{e,f}, p_{t+1}^{e,s})$. $H$ has a unique interior fixpoint :

$$\bar{p} = H(\bar{p}, \bar{p}) \Rightarrow \bar{p} = M.$$

At the steady state where $p_{t+1}^{e,f} = p_{t+1}^{e,s} = \bar{p}$ the partial derivatives are :

$$H_1(\bar{p}, \bar{p}) < \mu \text{ and } H_2(\bar{p}, \bar{p}) < (1 - \mu).$$

Thus the steady state is locally stable under learning. If there was only one type of learner such that $\mu = 1$ the condition for local stability $F'(\bar{p}) < 1$ together with $\Sigma_t a_t = \infty$ ensures global stability under learning, see chapter 2. □

Let us return to the model, and make the following assumption on the function H.

**Assumption 5.3.** The function $H$ is strictly concave function in $p_{t+1}^{e,f}$, $p_{t+1}^{e,s}$.

This assumption on $H$ is not trivial, for example, if $\sigma > 1$ the function $H$ defined in (5.17) does not satisfy assumption 5.3 nor assumption 5.1. The case where $\sigma > 1$ can be used to study cycles. Assumption 5.3 is an attempt to derive some properties on the function $H$. If for example $\mu = 1$, then $H$ is reduced to $F$. In addition if $\sigma < 1$ such that assumption 5.3 is satisfied, this results in the sequence $\{p_{t+1}^{e,f}\}_t$ being decreasing or increasing during the movement to a steady state, depending on the initial condition being either above or below the steady state. However when $0 < \mu < 1$ the sequences might not be monotone as shown in figures 5.1b, 5.1c, 5.2b, 5.2c and figure 5.3, but for $t$ sufficiently large the sequences become monotone and converge to the steady state.

If $p_{t+1}^{e,f}, p_{t+1}^{e,s} > \bar{p}$ then $p_t = H(p_{t+1}^{e,f}, p_{t+1}^{e,s}) > \bar{p}$, or if $p_{t+1}^{e,f}, p_{t+1}^{e,s} < \bar{p}$ then $p_t = H(p_{t+1}^{e,f}, p_{t+1}^{e,s}) < \bar{p}$, since $H$ is is increasing in both arguments according to assumption 5.1. The concavity assumption on $H$ can be used to show the following properties. If $p_{t+1}^{e,s} > p_{t+1}^{e,f} > \bar{p}$ then

$$p_{t+1}^{e,s} > H(p_{t+1}^{e,f}, p_{t+1}^{e,s}) = p_t \tag{5.19}$$

The strict concavity of $H$ implies

$$H(p_{t+1}^{e,f}, p_{t+1}^{e,s}) - H(\bar{p}, \bar{p}) < H_1(\bar{p}, \bar{p})(p_{t+1}^{e,f} - \bar{p}) + H_2(\bar{p}, \bar{p})(p_{t+1}^{e,s} - \bar{p})$$
$$< \mu(p_{t+1}^{e,f} - \bar{p}) + (1-\mu)(p_{t+1}^{e,s} - \bar{p}) < (p_{t+1}^{e,s} - \bar{p}) \quad \Rightarrow$$

$$H(p_{t+1}^{e,f}, p_{t+1}^{e,s}) < p_{t+1}^{e,s}.$$

We cannot be certain that $p_{t+1}^{e,f} > H(p_{t+1}^{e,f}, p_{t+1}^{e,s})$ as well.

Furthermore, if $p_{t+1}^{e,s} < p_{t+1}^{e,f} < \bar{p}$ then

$$p_{t+1}^{e,s} < H(p_{t+1}^{e,f}, p_{t+1}^{e,s}). \tag{5.20}$$

Since H is strictly concave in $p_{t+1}^{e,f}, p_{t+1}^{e,s}$ and $H(p_{t+1}^{e,s}, p_{t+1}^{e,s}) = F(p_{t+1}^{e,s})$ then

$$H(\bar{p}, \bar{p}) - H(p_{t+1}^{e,f}, p_{t+1}^{e,s}) < H(\bar{p}, \bar{p}) - H(p_{t+1}^{e,s}, p_{t+1}^{e,s})$$

$$= H(\bar{p}, \bar{p}) - F(p_{t+1}^{e,s}) < \bar{p} - p_{t+1}^{e,s} \qquad \Rightarrow$$

$$p_{t+1}^{e,s} < H(p_{t+1}^{e,f}, p_{t+1}^{e,s}),$$

again it is not certain that $p_{t+1}^{e,f} < H(p_{t+1}^{e,f}, p_{t+1}^{e,s})$. If $p_{t+1}^{e,i} > \bar{p} > p_{t+1}^{e,j}$ then

$$p_{t+1}^{e,i} > p_t > p_{t+1}^{e,j} \tag{5.21}$$

for $i,j = 1,2$ and $i \neq j$ according to (5.19) and (5.20).

We can now show that the steady state is globally stable under learning given assumption 5.3, this will follow from (5.19)-(5.21). Proposition 5.2 describes the case when the forecasts for the fast and slow learner are above steady state. In proposition 5.3 the forecasts are below the steady state. The other cases can be covered by these two situations as explained below.

**Proposition 5.2.** *Given assumptions 5.1-5.3. If* $p_1^{e,s} = p_1^{e,f} > \bar{p}$ *and* $p_{t+1}^{e,f} > \bar{p}$, $p_{t+1}^{e,s} > \bar{p}$ *for all t, then*

*(i)* $p_t > \bar{p}$ *and* $p_{t+1}^{e,f} < p_{t+1}^{e,s}$ *for all* $t \geq 1$

*(ii)* $\{p_{t+1}^{e,f}\}_t$ *and* $\{p_{t+1}^{e,s}\}_t$ *are convergent sequences with* $lim_{t\to\infty}\, p_{t+1}^{e,f} = \bar{p}$ *and* $lim_{t\to\infty}\, p_{t+1}^{e,s} = \bar{p}$.

*(iii)* $\{p_t\}_t$ *is a convergent sequence, with* $lim_{t\to\infty}\, p_t = \bar{p}$.

**Proof.** (i) Since $p_{t+1}^{e,f} > \bar{p}$, $p_{t+1}^{e,s} > \bar{p}$ for all $t$ then $p_t = H(p_{t+1}^{e,f}, p_{t+1}^{e,s}) > H(\bar{p}, \bar{p}) = \bar{p}$ for all t.

$\mathbf{S}^{\mathbf{o}}$ : At $t = 1$ $p_1^{e,f} = p_1^{e,s} > p_0$, and $a_s < a_f$ :

$$p_2^{e,f} = p_1^{e,f} + a_f(p_0 - p_1^{e,f}) < p_1^{e,f} + a_s(p_0 - p_1^{e,f})$$

$$= p_1^{e,s} + a_s(p_1^{e,s} - p_0) = p_2^{e,s}$$

$\mathbf{T}^{\mathbf{0}}$ : Assume that $p_{t+1}^{e,f} < p_{t+1}^{e,s}$ then

$$p_{t+2}^{e,f} = p_{t+1}^{e,f} + a_f(p_t - p_{t+1}^{e,f}) = (1 - a_f)\,p_{t+1}^{e,f} + a_f p_t < (1 - a_f)\;p_{t+1}^{e,s} + a_f p_t$$

$$= p_{t+1}^{e,s} + a_f(p_t - p_{t+1}^{e,s}) < p_{t+1}^{e,s} + a_s(p_t - p_{t+1}^{e,s}) = p_{t+2}^{e,s}.$$

where the last inequality is due to $p_t - p_{t+1}^{e,s} < 0$ from (19) and $a_s < a_f$.

(ii) Since $p_1^{e,f} = p_1^{e,s} > \bar{p}$, then $p_0 = H(p_1^{e,f}, p_1^{e,f}) > \bar{p}$ according to (i). From the strict concavity of $H$ : $p_0 < p_1^{e,f}$. From the learning rules we have $p_2^{e,f} < p_1^{e,f}$ and $p_2^{e,s} < p_1^{e,s}$. The price at time 1 is less than the price at time 0, $p_1 < p_0$, since the forecasts $p_2^{e,f}$ and $p_2^{e,s}$ has decreased. Furthermore, from (i) $p_2^{e,s} > p_2^{e,f}$, and $p_2^{e,s} > p_1$ from (19).

At time $t$ we have $p_t^{e,s} > p_t^{e,f}$ and $p_t^{e,s} > p_{t-1}$ such that $p_{t-1} - p_t^{e,s} < 0$ and

$$p_{t+1}^{e,s} = p_t^{e,s} + a_s(p_{t-1} - p_t^{e,s}) < p_t^{e,s},$$

and $\{p_t^{e,s}\}_t$ is a decreasing sequence that is bounded below by $\bar{p}$. Since $p_t^{e,s} > p_t^{e,f}$ from (i) and $p_t^{e,f} > \bar{p}$ by assumption, the sequence $\{p_t^{e,f}\}_t$ has to decrease when t is sufficiently large, although it might not be decreasing initially. Thus $p_t^{e,f} > p_t$ and.

$$p_{t+1}^{e,f} = p_t^{e,f} + a_f(p_{t-1} - p_t^{e,f}) < p_t^{e,f}$$

for $t$ sufficiently large. Now choose an integer $N_1$ sufficiently large such that $p_{t+1}^{e,f}$ and $p_{t+1}^{e,s}$ for $t > N_1$ are decreasing. Since both sequences are bounded below by $\bar{p}$, the two sequences converges and $\lim_{t\to\infty} p_{t+1}^{e,f} \geq \bar{p}$ and $\lim_{t\to\infty} p_{t+1}^{e,s} \geq \bar{p}$. Let $\lim_{t\to\infty} p_{t+1}^{e,f} = p^*$ and $\lim_{t\to\infty} p_{t+1}^{e,s} = q^*$. We have to show that $p^* = \bar{p}$ and $q^* = \bar{p}$. This is done by a contradiction. Let us assume that $p^* > \bar{p}$ and $q^* \geq \bar{p}$. Let $d_1 = H(p^*,q^*) - p^* < 0$ and $d_2 = H(p^*,q^*) - q^* < 0$. When $t > N_2 > N_1$ then

$$p_{t+1}^{e,f} = p_t^{e,f} + a_f(H(p_t^{e,f}, p_t^{e,s}) - p_t^{e,f}) \leq p_t^{e,f} + a_f d_1$$

$$p_{t+1}^{e,s} = p_t^{e,s} + a_s(H(p_t^{e,f}, p_t^{e,s}) - p_t^{e,s}) \leq p_t^{e,s} + a_s d_2,$$

and for all $r \geq 1$ :

$$p_{t+r}^{e,f} < p_t^{e,f} + \sum_{i=1}^{r} a_f d_1 \quad \text{and} \quad p_{t+r}^{e,s} < p_t^{e,s} + \sum_{i=1}^{r} a_s d_2 .$$

If we can choose $r$ sufficiently large we have a contradiction to $p^* > \overline{p}$ and $q^* > \overline{p}$, because $a_f$ , $a_s$ are positive and $d_1$ , $d_2$ are negative. Hence the expected prices converge to the steady state $\overline{p}$.

(iii) Since $H(.,.)$ is continuous, $\{p_t\}_t$ is a convergent sequence with $\lim_{t\to\infty} p_t = \overline{p}$.

■

**Remark.** We can illustrate proposition 5.2 in figure 5.1a.

Figure 5.1a about here. [1]

If $p_1^{e,f} > p_1^{e,s} > \overline{p}$ then the assumption in proposition 5.2 is not satisfied and (i) is not satisfied for the first periods, an example of this is shown in figure 5.1b. In figure 5.1b, $p_2^{e,s} < p_3^{e,s}$, but for $t \geq 4$ $p_{t+1}^{e,s}$ is decreasing and the both sequences converges. The increase in $p_t^{e,s}$ from $t = 2$ to $t = 3$ is due to high values of $p_1^{e,f}$ and $\mu$ giving a high value of $p_1$ such that $p_1 > p_2^{e,s}$. We have $p_2^{e,f} > p_1$ from (5.19). However, at some time $t$ $p_{t+1}^{e,s} > p_{t+1}^{e,f}$ because the fast learners have a higher speed of convergence and from (5.19) we have $p_{t+1}^{e,s} > H(p_{t+1}^{e,f}, p_{t+1}^{e,s})$. Thus we can use proposition 5.2 and show convergence.

Figure 5.1b about here.

[1] We use the C.E.S-functions with the parameter values $\mu = 0.5$, $\varepsilon = 0.2$, $\sigma = 0.41$, $a_f = 0.2$, $a_s = 0.05$ to generate figure 1.a and the following figures, but the initial conditions and money supply varies. In figure 1.a $p_1^{e,f} = p_1^{e,s} = 30$ and M = 15.

If we look at the case where $p_1^{e,s} > p_1^{e,f} > \bar{p}$, the we can find parameter values such that $p_{t+1}^{e,f}$ is increasing in the first few periods, this is illustrated in figure 5.1c. In figure 5.1c we use the parameter values from figure 5.1a but change the initial conditions and $\mu$. However for t sufficiently large we can use proposition 5.2.

Figure 5.1c about here

It is from (ii) in proposition 5.2, we have the terminology of fast and slow learners, since the fast agents adjust their expectations more rapidly than slow agents such that for *t* is sufficient large :

$$|p_{t+1} - p_{t+1}^{e,f}| < |p_{t+1} - p_{t+1}^{e,s}|$$

The fast agents are not learning "more", since they still use the same learning rule. The case where the initial price expectations below the steady state price is described in proposition 5.3.

**Proposition 5.3.** *Given assumptions 5.1-5.3. If* $p_1^{e,f} = p_1^{e,s} < \bar{p}$ *and* $p_{t+1}^{e,f} < \bar{p}$, $p_{t+1}^{e,s} < \bar{p}$ *for all t, then*

*(i)* $p_t < \bar{p}$ *and* $p_{t+1}^{e,f} > p_{t+1}^{e,s}$ *for all* $t \geq 1$.

*(ii)* $\{p_{t+1}^{e,f}\}_t$ *and* $\{p_{t+1}^{e,s}\}_t$ *are convergent sequences with* $lim_{t\to\infty} p_{t+1}^{e,f} = \bar{p}$ *and* $lim_{t\to\infty} p_{t+1}^{e,s} = \bar{p}$.

*(iii)* $\{p_t\}_t$ *is a convergent sequence, with* $lim_{t\to\infty} p_t = \bar{p}$.

**Proof.** This is similar to the proof of proposition 5.2. ■

**Remark.** We can illustrate proposition 5.3 in figure 5.2a. In figure 5.2a $\{p_{t+1}^{e,f}\}_t$ and $\{p_{t+1}^{e,s}\}_t$ are monotonically increasing such that $\{p_t\}_t$ is monotonically increasing.

Figure 5.2a about here.

If $p_1^{e,f} < p_1^{e,s} < \overline{p}$ then $p_{t+1}^{e,s}$ is not increasing initially and (i) is not satisfied as shown in figure 5.2b. However for t sufficiently large (i) is satisfied and both sequences converge

Figure 5.2b about here.

If $p_1^{e,s} < p_1^{e,f} < \overline{p}$, we can find parameter values such that $p_{t+1}^{e,f}$ is not monotonically increasing see figure 2c for an example, but for t sufficiently large we can use proposition 3 again and we have convergence.

Figure 5.2c about here.

We use the same parameter values as in figure 5.2b, but change the initial conditions and the fraction of learners $\mu$. In the propositions above we studied the cases where the initial forecast of the fast and slow learners were either above or below the steady state. We still need to analyse the cases where :

$$p_1^{e,f} > \overline{p} > p_1^{e,s} \text{ and } p_1^{e,f} < \overline{p} < p_1^{e,s},$$

Look at the case where the fast learner initially are above and the slow learner is initially below, $p_1^{e,f} > \overline{p} > p_1^{e,s}$, the other situation is treated symmetrically, then we cannot tell whether the actual price initially $p_0$ is above or below the steady state, but from (21) we have

$$p_1^{e,f} > p_0 > p_1^{e,s}$$

An example of this situation is shown in figure 5.3, where the fast learners initially are above the steady state in the first periods, but decrease and fall below the steady state. In this case we can use proposition 5.3, and the sequences converge to the steady state. It is not possible that the economy would converge to a cycle or a sunspot, since assumption 1 ensures that H is increasing in both arguments, and the strict concavity of U and strict convexity of V ensures a unique interior steady state.

Figure 5.3 about here.

It is possible that $p_{t+1}^{e,f} > p_t > p_{t+1}^{e,s}$ for all $t$ then we would use a combination of proposition 5.2 and 5.3 to show convergence. Thus the steady state is stable under learning, and depending on the initial conditions we can use either proposition 5.2 or 5.3.

Figure 5.3 about here.

It is possible that $p_{t+1}^{e,f} > p_t > p_{t+1}^{e,s}$ for all $t$ then we would use a combination of proposition 5.2 and 5.3 to show convergence. Thus the steady state is stable under learning, and depending on the initial conditions we can use either proposition 5.2 or 5.3.

### 5.4. The welfare comparison.

When we compare the level of welfare between fast and slow learners, we have to calculate the consumption for each of the agents and insert this into the welfare function. The welfare function is given by $U(c) - V(n)$ for both fast and slow learners. The labour supply is given by equation (5.3). If we insert (5.7) into these equations, we have

$$n_t^f = \xi\left(\frac{H(p_{t+1}^{e,f}, p_{t+1}^{e,s})}{p_{t+1}^{e,f}}\right) \quad \text{and} \quad n_t^s = \xi\left(\frac{H(p_{t+1}^{e,f}, p_{t+1}^{e,s})}{p_{t+1}^{e,s}}\right).$$

At time $t + 1$ the fast learner's actual consumption is given by:

$$p_{t+1} c_{t+1}^f = M_t^f = p_t n_t^f \qquad \Rightarrow \qquad c_{t+1}^f = \frac{p_t}{p_{t+1}} n_t^f .$$

The slow learner's actual consumption is determined the same way: $c_{t+1}^s = (p_t/p_{t+1})$ $n_t^s$. Since $p_{t+1}$ , $p_t$, $n_t^f$ and $n_t^s$ are functions of $(p_{t+1}^{e,f}, p_{t+1}^{e,s})$ the actual consumption $c_{t+1}^f$ and $c_{t+1}^s$ are functions of $(p_{t+1}^{e,f}, p_{t+1}^{e,s})$ as well. With the actual consumption and actual labour supply we calculate the welfare. Given these values, we have the *ex post* welfare sequences $\{U(c_{t+1}^f) - V(n_t^f)\}_{t=0}^{\infty}$ and $\{U(c_{t+1}^s) - V(n_t^s)\}_{t=0}^{\infty}$.

Let us assume that the economy initially is above the steady state and the fast and slow learner have the same initial forecast : $p_1^{e,f} = p_1^{e,s} > \bar{p}$. From proposition 2 $p_{t+1}^{e,f} < p_{t+1}^{e,s}$ for all $t \geq 1$, and the fast learners labour supply and actual consumption are larger than the slow learners labour supply and consumption:

$$n_t^f = \xi(p_t/p_{t+1}^{e,f}) > \xi(p_t/p_{t+1}^{e,s}) = n_t^s \qquad \text{for all } t \geq 1$$

$$c_{t+1}^f = (p_t/p_{t+1})\, n_t^f > (p_t/p_{t+1})\, n_t^s = c_{t+1}^s \qquad \text{for all } t \geq 1 ,$$

because $\xi$ is an increasing function. The expected real interest rate for the slow learners $p_t / p_{t+1}^{e,s} < 1$ for all t, the fast learners expected real interest rate $p_t / p_{t+1}^{e,f} < 1$ when $t$ is sufficiently large, because $p_{t+1}^{e,f} > p_t$ for $t$ sufficiently large. Since $\xi$ is increasing, the labour supply in the steady state is higher than the slow learners labour supply :

$$\bar{n} = \xi(1) > \xi(p_t / p_{t+1}^{e,s}) = n_t^s \quad \text{for all } t \geq 1.$$

$$\bar{c} > c_{t+1}^s \quad \text{for all } t \geq 1.$$

For the fast learner we have $\bar{n} = \xi(1) > \xi(p_t / p_{t+1}^{e,f}) = n_t^f$ and $\bar{c} > c_{t+1}^f$ for $t$ sufficiently large. We can now compare the welfare between two learners born at time $t$.

**Proposition 5.4.** *Given assumptions 5.1-5.3. Let* $\{n_t^s, c_{t+1}^s\}_{t=0}^{\infty}$ *and* $\{n_t^f, c_{t+1}^f\}_{t=0}^{\infty}$ *denote the slow learners and fast learners labour supply and consumption in a temporary equilibrium. If* $p_1^{e,f} = p_1^{e,s} > \bar{p}$, *then*

*(i) At* $t = 0$ *the welfare is equal* $U(c_1^s) - V(n_0^s) = U(c_1^f) - V(n_0^f)$.

*(ii) For t sufficiently large the fast learners are better off than the slow learners:*

$$U(c_{t+1}^f) - V(n_t^f) > U(c_{t+1}^s) - V(n_t^s).$$

**Proof.** (i) When $t = 0$, $p_1^{e,f} = p_1^{e,s}$ hence $n_0^f = n_0^s$ and $c_1^f = c_1^s$ such that the welfare is equal.

(ii) When $t$ is suffiently large, $\bar{c} > c_{t+1}^f > c_{t+1}^s$ and since $U$ is strictly concave:

$$U(c_{t+1}^f) - U(c_{t+1}^s) > U'(c_{t+1}^f)(c_{t+1}^f - c_{t+1}^s) > U'(\bar{c})(c_{t+1}^f - c_{t+1}^s) \tag{5.22}$$

Note that we use $U'(c_{t+1}^f)$ in the first inequality. The first-order condition at the steady state is given by $U'(\bar{c}) = V'(\bar{n})$, hence the right-hand side of (5.22) is equal to:

$$= V'(\bar{n})(\frac{p_t}{p_{t+1}} n_t^f - \frac{p_t}{p_{t+1}} n_t^s) \tag{5.23}$$

since $V$ is strictly convex and $\bar{n} > n_t^f > n_t^s$, (5.23) is greater than:

$$> \left\{ \frac{V(n_t^f) - V(n_t^s)}{n_t^f - n_t^s} \right\} \frac{p_t}{p_{t+1}} (n_t^f - n_t^s) = ( V(n_t^f) - V(n_t^s) ) \frac{p_t}{p_{t+1}}$$

$$> V(n_t^f) - V(n_t^s)$$

because $p_t/p_{t+1} > 1$ $t$ sufficiently large. Hence

$$U(c_{t+1}^f) - U(c_{t+1}^s) > V(n_t^f) - V(n_t^s) \Rightarrow$$

$$U(c_{t+1}^f) - V(n_t^f) > U(c_{t+1}^s) - V(n_t^s). \quad \blacksquare$$

**Remark.** The result in proposition 5.4 is shown in figure 5.4a, note that in figure 5.4a the fast learners are better off for all $t > 1$ and not just when $t$ is sufficiently large. A situation where the slow learners are better off in the first periods is shown in figure 5.4b, but the proposition still holds for $t$ sufficiently large. In figure 5.4b the initial conditions are $p_1^{e,f} > p_1^{e,s} > \bar{p}$.

Figure 5.4a about here.

Figure 5.4b about here

The intuition behind this result is that forecast made by the fast learner born at time $t$, $p_{t+1}^{e,f}$, is "closer" to the actual the actual price $p_t$ compared to the slow agent's forecast $p_{t+1}^{e,s}$, and the fast agent is closer to the actual optimum. Thus although the fast agent does not have rational expectations, he does produce a better forecast than the slow agent. If we only had $\varepsilon$ fast learners, where $\varepsilon$ is "small" but positive, then fast learners would still be better off, since this small number of fast learners would still benefit from making a better forecast. An extension of the model would be to introduce a perfect foresight agent in the model and let the rest of the agents be learners. The perfect foresight agent will benefit from the fact that all the other agents are learners, because the perfect agent know that actual price and is solving the true optimisation problem. This should not be surprising since the perfect foresight agent know the path of equilibrium prices, while the learners are making a wrong forecast.

In the second case, where the initial expected prices are lower than the steady state price, we can show the same result. Let us assume that assumptions 5.1-5.3 are satisfied and $p_1^{e,f} = p_1^{e,s} < \bar{p}$. The labour supply of the fast learners are less than the slow learners labour supply :

$$n_t^f = \xi(p_t / p_{t+1}^{e,f}) < \xi(p_t / p_{t+1}^{e,s}) = n_t^s \quad \text{for all } t \geq 1,$$

because $\xi$ is an increasing function in the expected interest rate (assumption 5.1) and $p_{t+1}^{e,f} > p_{t+1}^{e,s}$ according to proposition 5.3. The consumption of fast learners is less than slow learners consumption :

$$c_{t+1}^f = (p_t/p_{t+1})\, n_t^f < (p_t/p_{t+1})\, n_t^s = c_{t+1}^s \quad \text{for all } t \geq 1.$$

The steady state are less than the labour supply at time $t$ and consumption at time $t$+1

$$\bar{n} < n_t^s \quad , \quad \bar{c} < c_{t+1}^s \quad \text{for all } t \geq 1.$$

$$\bar{n} < n_t^f \quad , \quad \bar{c} < c_{t+1}^f \quad \text{for } t \text{ sufficiently large.}$$

We have the following proposition regarding the welfare of the fast and slow learners.

**Proposition 5.5.** *Given assumptions 5.1-5.3. Let* $\{n_t^s, c_{t+1}^s\}_{t=0}^{\infty}$ *and* $\{n_t^f, c_{t+1}^f\}_{t=0}^{\infty}$ *denote the slow learners and fast learners labour supply and consumption in a temporary equilibrium. If* $p_1^{e,f} = p_1^{e,s} < \bar{p}$, *then at* $t = 0$ : $U(c_1^s) - V(n_0^s) = U(c_1^f) - V(n_0^f)$ *and for t sufficiently large the fast learners are better off than the slow learners :* $U(c_{t+1}^s) - V(n_t^s) < U(c_{t+1}^f) - V(n_t^f)$.

**Proof.** See appendix. ■

The result here is again due to the fast learners making a "better" forecast than the slow learners. When $p_1^{e,f} \neq p_1^{e,s}$ but $p_1^{e,f}$, $p_1^{e,s} < \bar{p}$ then proposition 5.5 might not hold initially, however for $t$ sufficiently large the proposition is still valid. If we look at the case where the initial conditions are given by $p_1^{e,s} > \bar{p} > p_1^{e,f}$, an example of this is shown figure 5.5, we have a similar result as in figure 5.4b. The slow agents are better off initially and for some periods ahead, but the fast learners overtake the slow learners and remain better off during the movement to the steady state see figure 5.5.

Figure 5.5 about here.

The reason for the slow being better off than the fast in the short run is that the slow are "closer" to the actual price. When $t$ is sufficient large the effect from the fast learners higher speed of learning will make his forecast closer to the actual price, and the fast learner have a higher welfare compared to the slow learners as before for all $t$ sufficently large.

### 5.5. The effect of a change in the money supply.

We assumed that the money supply $M$ was constant. If the government changes the money supply by printing more money, the steady state price $\overline{p}$ change. If $M$ is increased to $\widetilde{M}$ then it is easy to see that the steady state price level is increased as well, from (5.8) we have:

$$\frac{\widetilde{M}}{p_t} = \mu\xi\left(\frac{p_t}{p_{t+1}^{e,f}}\right) + (1-\mu)\xi\left(\frac{p_t}{p_{t+1}^{e,s}}\right).$$

Since the money supply has changed, $\overline{p}$ does not satisfy the above equation, the new steady state price is given by $\widetilde{p}\,\xi(1) = \widetilde{M}$. Since $\widetilde{M} > M$ then $\widetilde{p} > \overline{p}$ and the increase in the money supply increase the steady state price level, not surprisingly. There is no real effect the steady state labour supply $\widetilde{n} = \overline{n}$ and the steady state consumption $\widetilde{c} = \overline{c}$. The steady state welfare is unchanged and $U(\widetilde{c}) - V(\widetilde{n}) = U(\overline{c}) - V(\overline{n})$.

If we assume the economy is in the steady state $\overline{p}$ and the money supply is $M$. At time $T$ the government increase the money supply from $M$ to $\widetilde{M}$. If this is unanticipated by the agents born at time $T$ the expected prices at time are $p_{T+1}^{e,f} = p_{T+1}^{e,s} = \overline{p}$, since $p_{T+1}^{e,f}$, $p_{T+1}^{e,s}$ depend on the expected prices time $T-1$, $p_T^{e,f}$ and $p_T^{e,s}$, and the actual price at time $T-1$, $p_{T-1}$, according to the learning rules. For example, the government could change the money supply after the agents have updated the expected price according to the learning rule. In this situation there is a delay before the change in the money supply affects the expected prices. However, the actual price at time $T$, $p_T$, increases because the change in $M$ affects the function $H(.,.,\mu)$.

**Lemma 5.2.** *Given assumption 5.1, the partial derivative of H w.r.t. M is positive.*

**Proof.** See appendix. ■

From lemma 5.2, we have that an increase in $M$ increases the price $p_T$. Since $p_T$ increases so does $p^{e,f}_{T+2}$ and $p^{e,s}_{T+2}$ according to the learning rules. We can use proposition 5.3 to determine the price paths, because the initial condition $p^{e,f}_{T+1}$ = $p^{e,s}_{T+1} = \overline{p} < \widetilde{p}$. The fast learners are better off than the slow learners during the movement from the old steady state to the new steady state according to proposition 5.5. The increase in the money supply benefits the fast learners. If we made the opposite experiment such that the government decreased the money supply, we could use proposition 5.2. Hence fast learners would still be better off than slow learners during the learning transition when $t$ is sufficiently large. Thus a contractionary or expansionary monetary policy always make the fast learners better off compared to the slow learners for $t$ sufficiently large, when the agents have the same initial forecast.

The forecast at time $T - 1$, $p^{e,f}_{T}$ and $p^{e,s}_{T}$, and the actual price at time $T - 1$, $p_{T-1}$, was based on the money supply being equal to $M$. If the government announce the change in money supply before the agents born at time $T$ update their price, it would be natural to allow the agents to change their forecast $p^{e,i}_{T+1}$ $i = s,f$ instead of updating according to learning rule. In this case the two types agents at time $T$ might have different initial conditions $p^{e,f}_{T+1} \neq p^{e,s}_{T+1}$. When $t > T$ the agents use the learning rules. In this situation the slow learners might be better off in the initial periods after the change, but for $t$ sufficently large the fast learners over take the slow learners.

## 5.6. The model with n different types of learners.

If we assume there are n different types of learners instead of two types of learners, this does not change the stability result and welfare results from the previous sections. The increase in number of types increase the number of difference equations in the dynamic system, but the results when $m = 2$ can be extended to the case with $m > 2$ different types. In section 2 we defined $p_t$ implict as function of $p_{t+1}^{e,1},..., p_{t+1}^{e,m}$ :

$$p_t = H(p_{t+1}^{e,1},..., p_{t+1}^{e,m}). \tag{5.24}$$

The agents learn again on the expected price according to the learning rule:

$$p_{t+1}^{e,i} = p_t^{e,i} + a_i (p_{t-1} - p_t^{e,i}) \quad \text{for all } t \text{ and } i = 1, \dots ,m. \tag{5.25}$$

Here $a_i$ denotes type $i$'s speed of learning and we assume

$$a_1 < a_2 < \dots < a_m .$$

The steady state $\overline{p}$ is determined by the equation $\overline{p} = H(\overline{p} ,...., \overline{p})$. If we substitute the temporary equilibrium condition (5.24) into the learning rules we have the following dynamic system consisting of $m$ non-linear difference equations:

$$\begin{aligned} p_{t+1}^{e,1} &= p_t^{e,1} + a_1 ( H(p_t^{e,1},..., p_t^{e,m}) - p_t^{e,1}) \quad \text{for all } t. \\ &\bullet \qquad\qquad\qquad\qquad \bullet \\ &\bullet \qquad\qquad\qquad\qquad \bullet \\ p_{t+1}^{e,m} &= p_t^{e,m} + a_m ( H(p_t^{e,1},..., p_t^{e,m}) - p_t^{e,m}) \quad \text{for all } t. \end{aligned} \tag{5.26}$$

The local stability of the steady state under learning is determined by the size of the eigenvalues of the Jacobian matrix for (5.26) at the point $(\overline{p} ,...., \overline{p})$ :

$$J(\overline{p} ,...., \overline{p}) = \begin{pmatrix} g_{11} \cdots\cdots\cdots\cdots g_{1m} \\ \\ g_{m1} \cdots\cdots\cdots\cdots g_{mm} \end{pmatrix}$$

where $g_{ij}$ is the partial derivative of $p_t^{e,i} + a_i ( H(p_t^{e,1},..., p_t^{e,m}) - p_t^{e,i})$ w.r.t. $p_t^{e,j}$ at $(\overline{p} ,...., \overline{p})$ for all $i, j = 1,...,m$. The partial derivatives $g_{ij}$ are given by

$$g_{ii} = 1 + a_i ( H_i(\overline{p} ,...., \overline{p}) - 1) \qquad \text{when } i = j$$

$$g_{ij} = a_i H_j(\overline{p},....,\overline{p}) \qquad \text{when } i \neq j,$$

where $H_i$ is the partial derivative of $H$ w.r.t. $p_t^{e,i}$. When $p_t^{e,i} = \overline{p}$ for all $i$, then

$$H_j(\overline{p},....,\overline{p}) = (\mu_j/\mu_i)H_i(\overline{p},....,\overline{p}) \qquad \text{for } i,j = 1,...,m.$$

The following theorem from Sydsaeter (1981) gives a condition for the eigenvalues of J to be less than one in absolute value.

**Theorem.** *Sydsaeter (1981). Let* $\mathbf{A} = [\, a_{ij} \,]$ *be an arbitrary* $m \times m$ *- matrix and assume that*

$$\sum_{j=1}^{m} \left| a_{ij} \right| < 1$$

*Then, all the eigenvalues of* $\mathbf{A}$ *have moduli less than 1.*

Since $H_i > 0$ for all $i = 1,...,$n the sum of the elements in the rows of $J$ has to be less than 1 :

$$g_{i1} + ..... + g_{im} < 1 \qquad \text{for } i = 1,..., m.$$

Let $i = 1$, then $g_{11} + ..... + g_{1m} < 1$ implies :

$$1 + a_1(H_1 - 1) + a_1 H_2 + .... + a_1 H_m < 1 \qquad \Rightarrow$$

$$H_1 + H_2 + .... + H_m < 1 \;\Rightarrow H_1 + \frac{\mu_2}{\mu_1} H_1 + .... + \frac{\mu_n}{\mu_1} H_1 < 1 \;\Rightarrow$$

$$\mu_1 H_1 + (1 - \mu_1)H_1 < \mu_1 \;\Rightarrow\; H_1 < \mu_1.$$

Since $H_i = (\mu_i/\mu_{j\cdot})H_j$ at the steady state $\overline{p}$, then

$$H_i = (\mu_i/\mu_{1\cdot})H_1 < \mu_i \quad \text{for all } i,$$

when $H_1 < \mu_1$. The condition for local stability under learning can be summarised in the following proposition.

**Proposition 5.6.** *Given assumptions 5.1 and 5.2. If there exists an* $i \in \{1,...,m\}$ *such that* $H_i < \mu_1$, *then the steady state is locally stable under learning.*

This is similar to proposition 1, if the $i$-th agent satisfies the stability condition , then the rest of the agents satisfies the condition as well. A possible extension of this result would be to let n go to infinity such that there is an infinite but countable number of different agents and

$$\sum_{i=1}^{\infty} \mu_i = 1.$$

The Jacobian matrix becomes a $\infty \times \infty$-matrix, and we cannot be sure that the theorem from Sydsaeter (1981) is valid for $m = \infty$. We could also introduce a continuum of agents, for example distributed over the unit interval or the unit ball. This is studied by Evans, Honkapohja and Marimon (1995).

Let us return to the global stability question and the welfare comparison, that is, will the steady state be globally stable under learning and will the welfare results still be valid when we have $m > 2$ types of agents. We extend assumption 3 on the function $H$ in order to cover the case with $m$ types of agents.

**Assumption 5.4.** The function $H$ is strictly concave in $p_1^{e,i}$ for all $i = 1,\ldots,m$.

Let us assume that the initial price expectations are above the steady state: $p_1^{e,1} = \ldots = p_1^{e,m} > \bar{p}$ . and concentrate on the case where $p_{t+1}^{e,i} > \bar{p}$ for all $t$ and $i = 1,\ldots,m$. In this case

$$p_t = H(p_{t+1}^{e,1},\ldots, p_{t+1}^{e,m}) > \bar{p} \text{ and}$$

There exists at least one $i$ such that $p_{t+1}^{e,i} > H(p_{t+1}^{e,1},\ldots, p_{t+1}^{e,m})$

With this property on the function $H$, then we have the following proposition on the expected prices and actual price.

**Proposition 5.7.** *Given assumptions 5.1-5.2 and 5.4. If* $p_1^{e,1} = \ldots = p_1^{e,m} > \bar{p}$ *and* $p_{t+1}^{e,i} > \bar{p}$ *for all t and i* = 1,...,*m. then*

*(i)* $p_{t+1}^{e,1} > \ldots > p_{t+1}^{e,m}$ *for all* $t \geq 1$.

*(ii)* $\{p_{t+1}^{e,i}\}_{t=0}^{\infty}$ *is a convergent sequence with* $\lim_{t\to\infty} p_{t+1}^{e,i} = \bar{p}$, *for all* $i = 1,...,m$.

*(iii)* $\{p_t\}_{t=0}^{\infty}$ *is a convergent sequence with* $\lim_{t\to\infty} p_t = \bar{p}$.

**Proof.** This is similar to the proof of proposition 5.2. ■

Since $p_{t+1}^{e,1} > ... > p_{t+1}^{e,m}$ for all $t$, the labour supply and the consumption can be ordered as follows:

$$n_t^1 < .... < n_t^m \text{ and } c_{t+1}^1 < ... < c_{t+1}^m \quad \text{for all } t \geq 1.$$

When $t$ is suffiently large we have $p_t / p_{t+1}^{e,i} < 1$ and the labour supply and consumption in the steady state are again larger than labour supply and consumption of the $i$-th agent:

$$\bar{n} > n_t^i \text{ and } \bar{c} > c_{t+1}^i \quad \text{when } t \text{ sufficiently large and } i = 1,...,m.$$

Hence we have a similar proposition to proposition 5.4.

**Proposition 5.8.** *Given assumptions 5.1-5.2 and 5.4. Let* $\{n_t^i, c_{t+1}^i\}_{t=0}^{\infty}$ *denote type i's supply of labour and consumption in a temporary equilibrium. If* $p_1^{e,1} = ... = p_1^{e,m} > \bar{p}$ *for all i,j* $= 1,...,m$ *then*

*(i) At* $t = 0$ *the welfare of the m different agents are equal :*

$$U(c_1^1) - V(n_0^1) = ..... = U(c_1^m) - V(n_0^m).$$

*(ii) When t is sufficiently large the welfare of the m different agents can be ranked :*

$$U(c_{t+1}^1) - V(n_t^1) > ..... > U(c_{t+1}^m) - V(n_t^m) \text{ for all } t \geq 1.$$

**Proof** . Similar to the proof of proposition 5.4. ■

If we assume that the initial expected prices are below the steady state price then we can prove similar propositions as propositions 5.3 and 5.5 but for $m$ different agents instead of 2 different agents. However, proposition 5.8 is not valid if the initial conditions are different as was the case when $m = 2$. If the initial conditions are different, then the slow learners might be better off initially and the periods ahead.

### 5.7. Conclusion.

We have studied a standard overlapping generations model, but extended it so that it includes 2 or $m$ types of agents. The condition for local stability of the steady state depended on the fraction of learners. Given a concavity assumption the steady state was globally stable. The concavity assumption together with an assumption on the initial condition on the agents forecast of the price could be used to prove that the fast learners had a higher welfare than the slow learners for $t$ sufficiently large, regardless of the initial conditions. This was due to higher speed of convergence such that the fast agents was close to the actual price. However, it was possible to show by simulations that the slow learners could be better off for some time during the learning transition to the steady state if the initial conditions were different.

The two types of learners did not change their speed of learning over time so there was no possibility for the slow learners to change their speed of learning. The agents are stuck with their learning rule. An possible extension of this would be to modify the learning rules so the agents take account of each others expectations. In this model the fraction of learners were exogenous and another extension would be to endogenise the fraction of learners. As a last case we extended the model to include $m$ types of agents. The local stability result and the welfare comparison was the same as in the case with two types, given the similar assumptions. We could try and introduce a continuum of agents as in Evans, Honkapohja and Marimon (1995). In this model there were no uncertainty, it is obvious to allow for uncertainty, for example, a shock to production or preferences. It could also be a monetary shock. In this situation the fast learners could have a constant speed of learning and the slow learners could have a decreasing speed of learning. The question is whether the economy would converge under learning, or if the noisy forecast produced by the fast learners cause the system to be unstable, but this is left for further investigation.

## Appendix to chapter 5.

*Proof of lemma 5.1.* For convenience look at the case where $m = 2$. If we differentiate equation (5.7) w.r.t. $p_{t+1}^{e,i}$, where $p_{t+1}^{e,j}$ $j \neq i$ and $\mu$ are kept constant:

$$\frac{\partial\left(\frac{M}{p_t}\right)}{\partial p_{t+1}^{e,i}} = \frac{\partial\left(\mu\xi(p_t / p_{t+1}^{e,i})\right)}{\partial p_{t+1}^{e,i}} + \frac{\partial\left((1-\mu)\xi(p_t / p_{t+1}^{e,j})\right)}{\partial p_{t+1}^{e,i}} \quad \Rightarrow$$

$$M\left(-\frac{1}{p_t^{\,2}}\right)\frac{\partial p_t}{\partial p_{t+1}^{e,i}} = \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,i}}\right)\left[\frac{1}{(p_{t+1}^{e,i})^2}\left(\frac{\partial p_t}{\partial p_{t+1}^{e,i}}p_{t+1}^{e,i} - p_t\right)\right]$$

$$+ (1-\mu)\xi'\left(\frac{p_t}{p_{t+1}^{e,j}}\right)\left[\frac{1}{p_{t+1}^{e,j}}\frac{\partial p_t}{\partial p_{t+1}^{e,i}}\right] \quad \Rightarrow$$

$$\frac{\partial p_t}{\partial p_{t+1}^{e,i}}\left(-\frac{M}{p_t^{\,2}} - \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,i}}\right)\frac{1}{p_{t+1}^{e,i}} - (1-\mu)\xi'\left(\frac{p_t}{p_{t+1}^{e,j}}\right)\frac{1}{p_{t+1}^{e,j}}\right)$$

$$= -\,\mu\xi'\left(\frac{p_t}{p_{t+1}^{e,i}}\right)\frac{p_t}{(p_{t+1}^{e,i})^2} \quad \Rightarrow$$

$$\frac{\partial p_t}{\partial p_{t+1}^{e,i}}\left(\frac{M}{p_t} + \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,i}}\right)\frac{1}{p_{t+1}^{e,i}} + (1-\mu)\xi'\left(\frac{p_t}{p_{t+1}^{e,j}}\right)\frac{1}{p_{t+1}^{e,j}}\right) = \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,i}}\right)\frac{p_t}{(p_{t+1}^{e,i})^2} \quad \Rightarrow$$

$$\frac{\partial p_t}{\partial p_{t+1}^{e,i}} = \frac{1}{\Lambda}\mu\xi'\left(\frac{p_t}{p_{t+1}^{e,i}}\right)\left(\frac{p_t}{p_{t+1}^{e,i}}\right)^2 > 0,$$

where $\Lambda = \left(\frac{M}{p_t} + \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,i}}\right)\frac{1}{p_{t+1}^{e,i}} + (1-\mu)\xi'\left(\frac{p_t}{p_{t+1}^{e,j}}\right)\frac{1}{p_{t+1}^{e,j}}\right) > 0$ and $\xi' > 0$

according to assumption 5.1. ■

*The condition* $H_2(\bar{p},\bar{p},\mu) = \frac{1-\mu}{\mu}H_1(\bar{p},\bar{p},\mu)$. From the proof of lemma 5.1, the partial derivatives are given by

$$\frac{\partial p_t}{\partial p_{t+1}^{e,f}} = \mu\frac{1}{\Lambda(p_{t+1}^{e,f}, p_{t+1}^{e,s}, \mu)}\xi'\left(\frac{p_t}{p_{t+1}^{e,f}}\right)\left(\frac{p_t}{p_{t+1}^{e,s}}\right)^2 \quad \text{and}$$

$$< \frac{V(n_t^s) - V(n_t^f)}{n_t^s - n_t^f} \frac{p_t}{p_{t+1}} (n_t^s - n_t^f) = ( V(n_t^s) - V(n_t^f) ) \frac{p_t}{p_{t+1}}$$

$$< V(n_t^f) - V(n_t^s)$$

because $p_t/p_{t+1} > 1$ for $t$ sufficiently large. Hence

$$U(c_{t+1}^s) - U(c_{t+1}^f) < V(n_t^s) - V(n_t^f) \qquad \Rightarrow$$

$$U(c_{t+1}^f) - V(n_t^f) > U(c_{t+1}^s) - V(n_t^s). \quad \blacksquare$$

*Proof of lemma 2.* Let us differentiate equation (5.7) w.r.t. $M$ where $p_{t+1}^{e,f}$, $p_{t+1}^{e,s}$ and $\mu$ all are kept constant:

$$\frac{\partial\left(\frac{M}{p_t}\right)}{\partial M} = \frac{\partial\left(\mu\xi(p_t / p_{t+1}^{e,f})\right)}{\partial M} + \frac{\partial\left((1-\mu)\xi(p_t / p_{t+1}^{e,s})\right)}{\partial M} \qquad \Rightarrow$$

$$\left(p_t - M\frac{\partial p_t}{\partial M}\right)\frac{1}{p_t^{\,2}} = \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,f}}\right)\frac{1}{p_{t+1}^{e,f}}\frac{\partial p_t}{\partial M} +$$

$$(1-\mu)\xi'\left(\frac{p_t}{p_{t+1}^{e,s}}\right)\left[\frac{1}{p_{t+1}^{e,s}}\frac{\partial p_t}{\partial M}\right] \Rightarrow$$

$$\frac{\partial p_t}{\partial M}\left(\frac{M}{p_t^{\,2}} + \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,f}}\right)\frac{1}{p_{t+1}^{e,f}} + (1-\mu)\xi'\left(\frac{p_t}{p_{t+1}^{e,s}}\right)\frac{1}{p_{t+1}^{e,s}}\right) = \frac{1}{p_t} \qquad \Rightarrow$$

$$\frac{\partial p_t}{\partial M}\left(\frac{M}{p_t} + \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,f}}\right)\frac{p_t}{p_{t+1}^{e,f}} + (1-\mu)\xi'\left(\frac{p_t}{p_{t+1}^{e,s}}\right)\frac{p_t}{p_{t+1}^{e,s}}\right) = 1 \qquad \Rightarrow$$

$$\frac{\partial p_t}{\partial M} = \frac{1}{\Lambda} > 0,$$

because $\xi' > 0$ according to assumption 5.1 then

$$\Lambda = \left(\frac{M}{p_t} + \mu\xi'\left(\frac{p_t}{p_{t+1}^{e,f}}\right)\frac{p_t}{p_{t+1}^{e,f}} + (1-\mu)\xi'\left(\frac{p_t}{p_{t+1}^{e,s}}\right)\frac{p_t}{p_{t+1}^{e,s}}\right) > 0. \quad \blacksquare$$

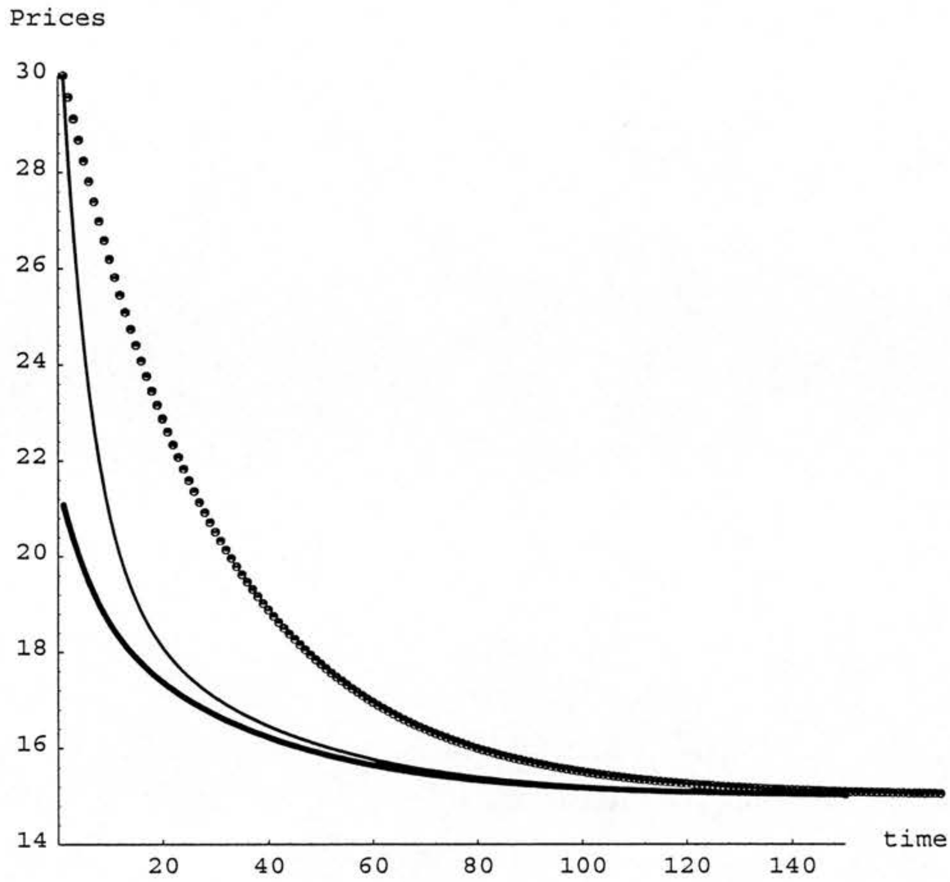



Figure 5.1a. The paths for prices, with $q_i^e = p_i^e > \bar{p} = 15$.
Slow learners expected price (dotted line),
fast learners expected price (straight line) and
the actual price (thick straight line).

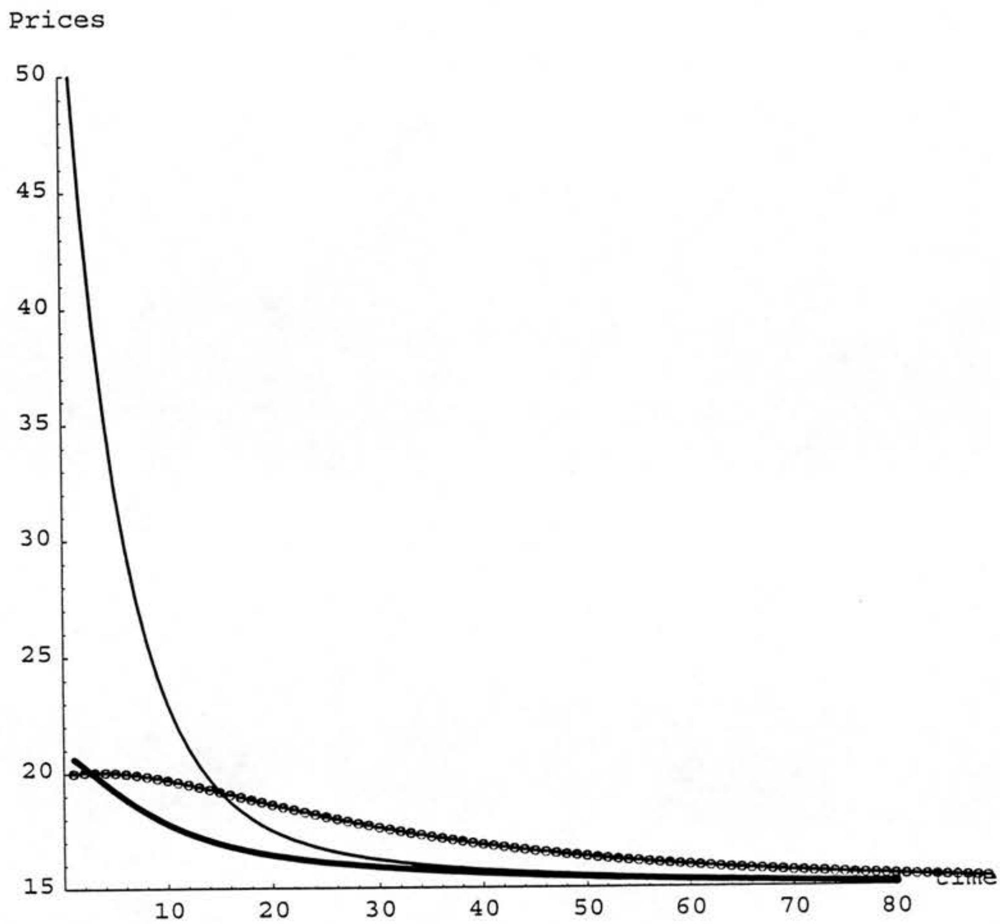



**Figure 5.1b. The paths for prices, where $p^e_1 = 50 > q^e_1 = 20 > \bar{p} = 15$.**
Slow learners expected price (dotted line), fast learners expected price (straight line) and the actual price (thick straight line).

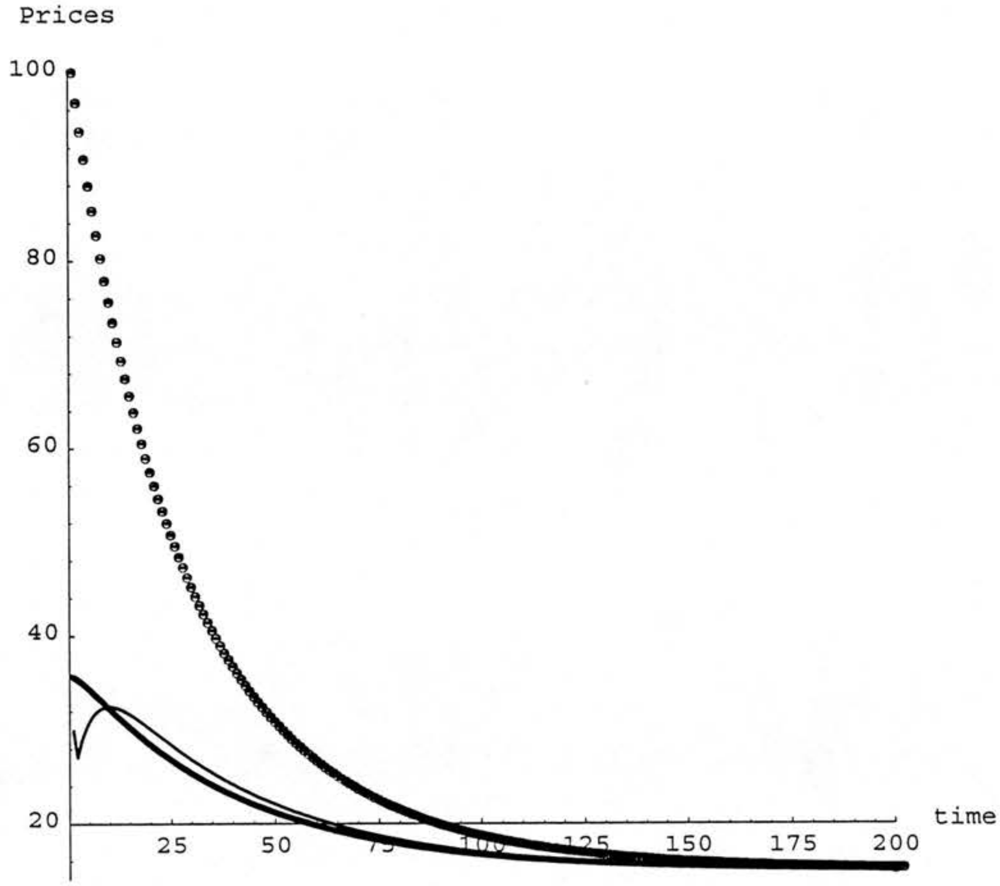



Figure 5.1c. The paths for prices, with $q_1^e = 100$, $p_1^e = 25 > \overline{p} = 15$ and $\mu = 0.05$.
The Slow learners expected price (dotted line), the fast learners expected price (straight line) and the actual price (thick straight line).

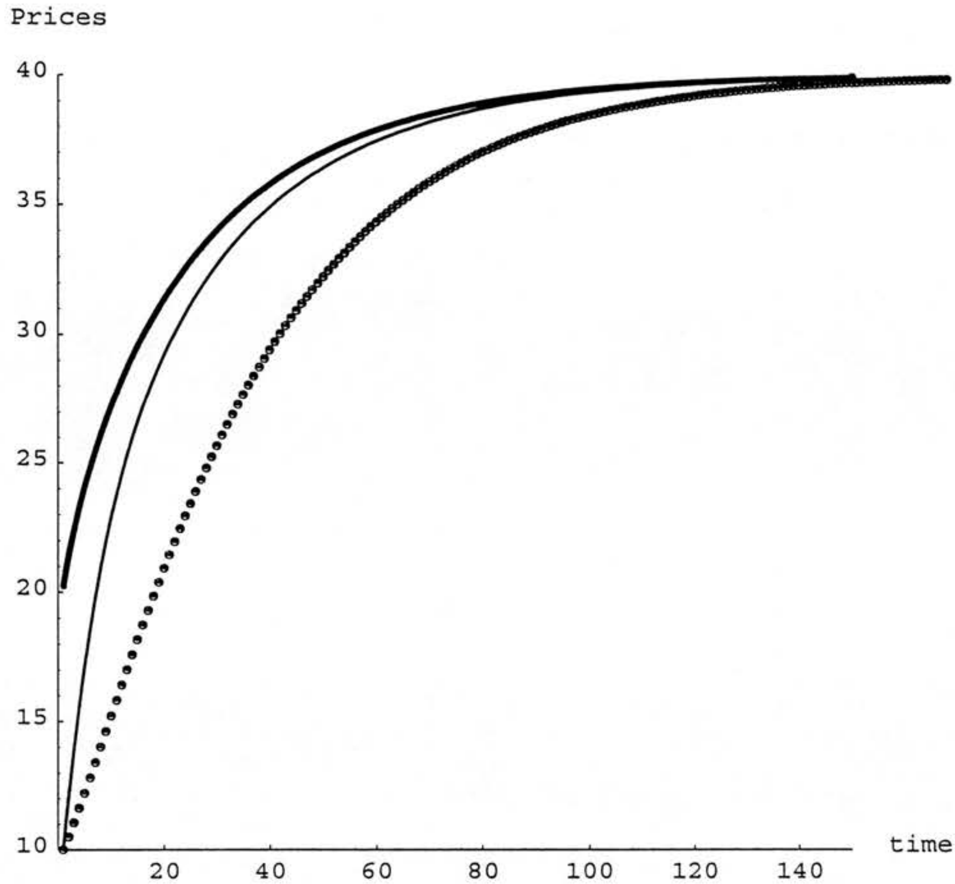



**Figure 5.2a.** The paths for prices, where $q_1^e = p_1^e < \bar{\bar{p}} = 40$.
Slow learners expected price (dotted line),
fast learners expected price (straight line) and
the actual price (thick straight line).

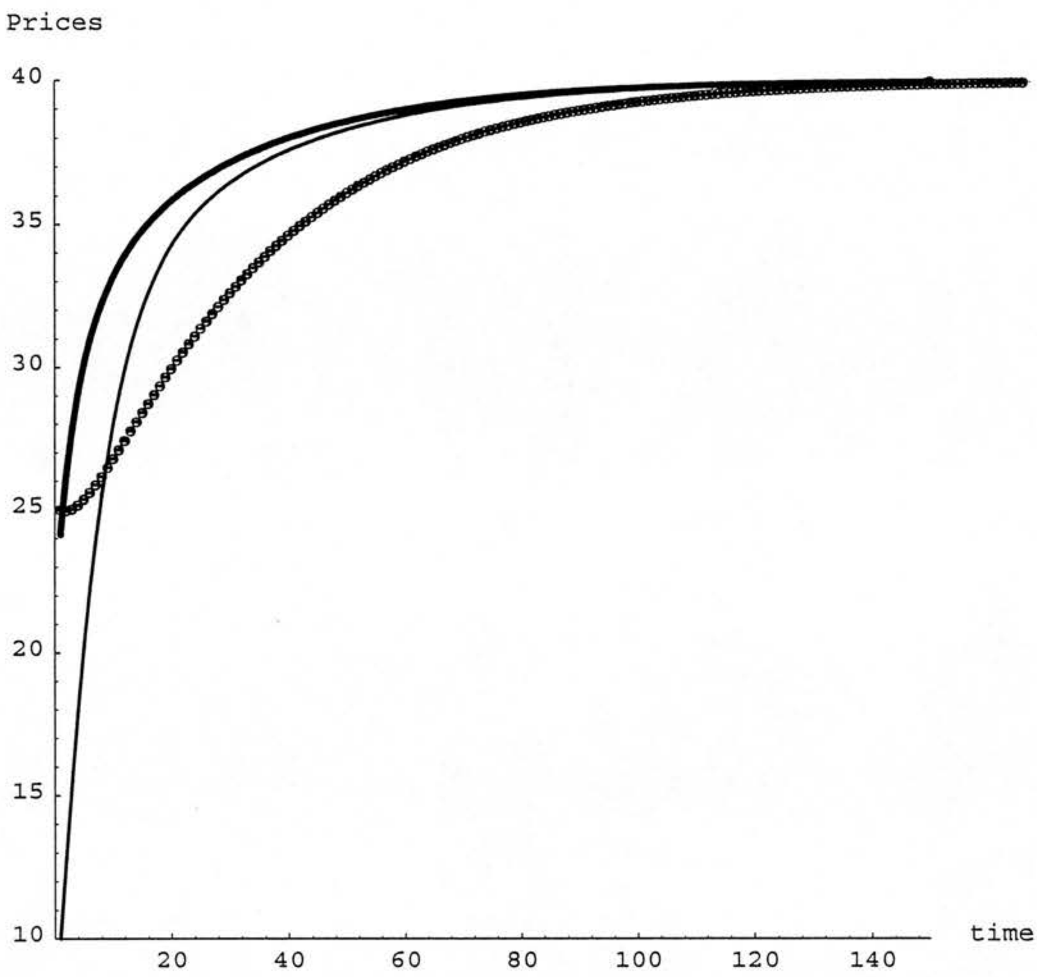



Figure 5.2b. The paths for prices,
where $p_1^e = 10 < q_1^e = 25 < \bar{p} = 40$.
Slow learners expected price (dotted line),
fast learners expected price (straight line) and
the actual price (thick straight line).

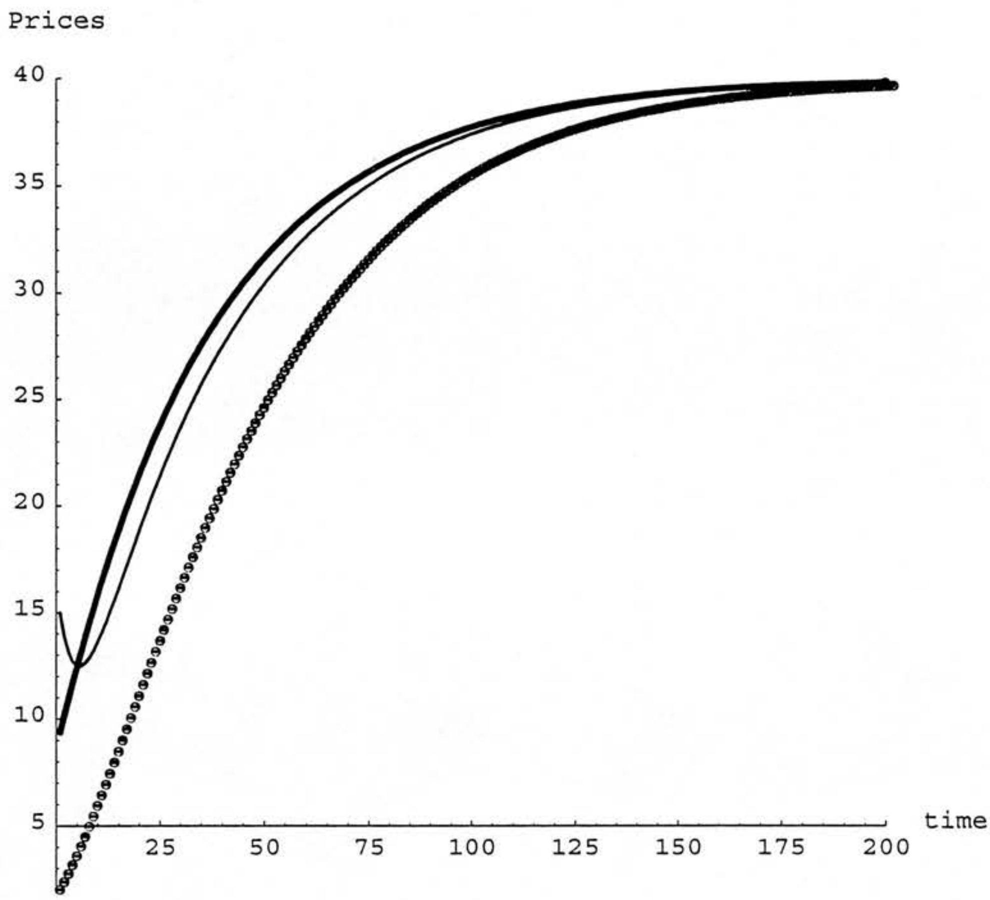



**Figure 5.2c. The paths for prices, where $q_1^e = 2 < p_1^e = 15 < \bar{p} = 40$ and $\mu = 0.05$. The slow learners expected price (dotted line), the fast learners expected price (straight line) and the actual price (thick straight line).**

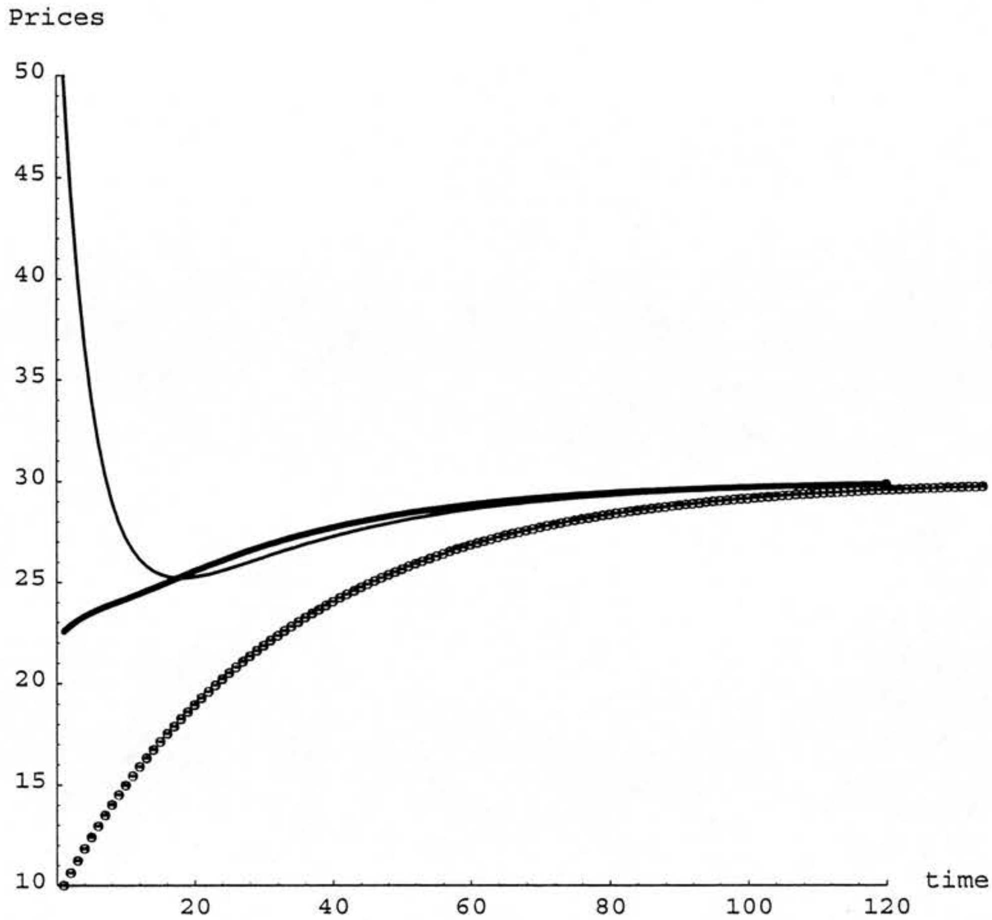



**Figure 5.3. The paths for prices, where $q_1^e = 10 < \bar{p} = 30 < p_1^e = 50$.**
**Slow learners expected price (dotted line), fast learners expected price (straight line) and the actual price (thick straight line).**

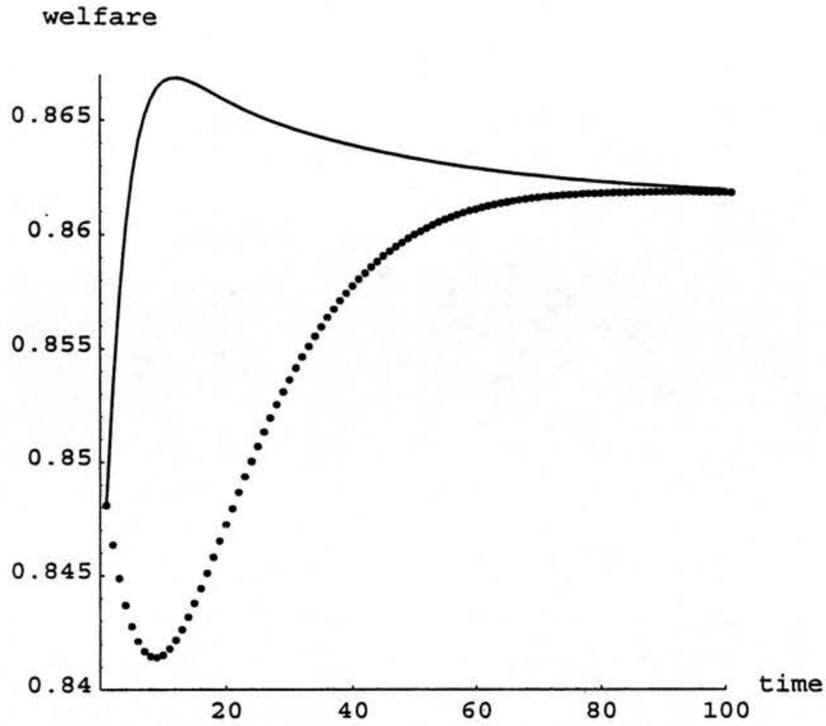



Figure 5.4a. The welfare paths, where $p_1^c = q_1^c > \bar{p}$.
The slow learners welfare (dotted line),
the fast learners welfare (straight line).

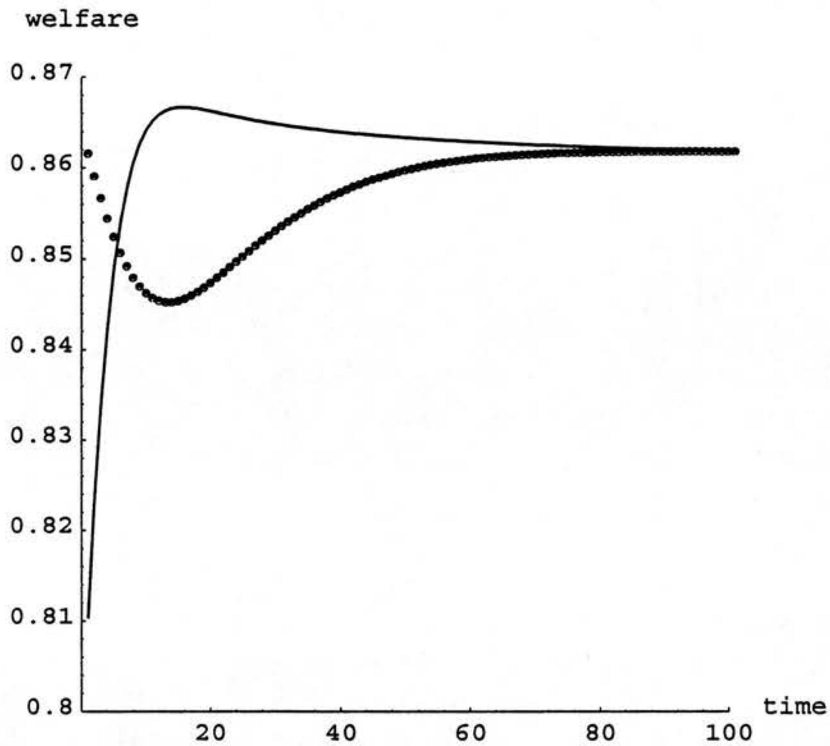



**Figure 5.4b. The welfare paths, where $p_1^c > q_1^c > \bar{p}$. The slow learners welfare (dotted line), the fast learners welfare (straight line).**

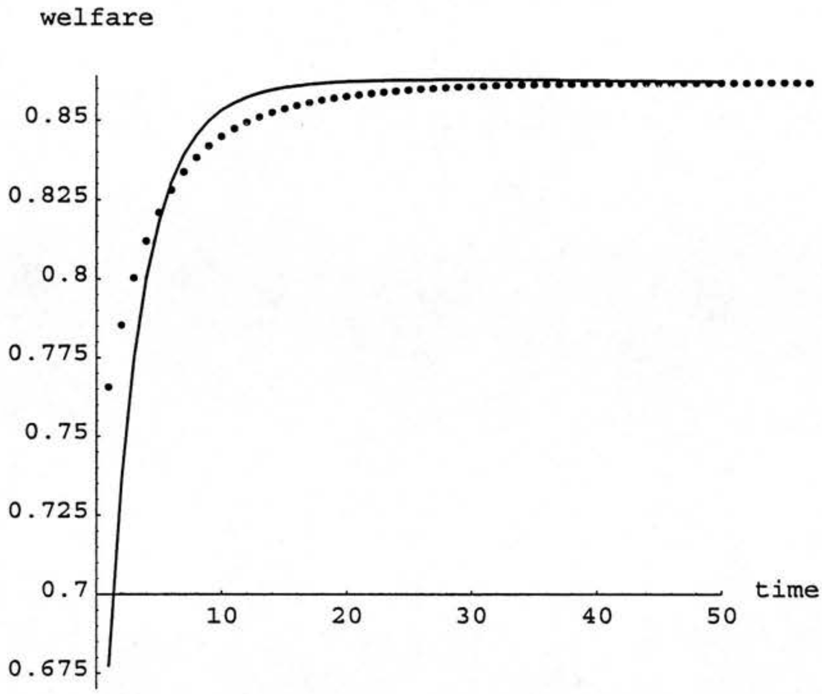



**Figure 5.5. The welfare paths, where $\bar{q}_1^c > \bar{p} > p_1^c$.
The slow learners welfare (dotted line),
The fast learners welfare (straight line).**

# Chapter 6. The local stability of a 2-cycle under learning.

## 6.1. Introduction.

There has been a strong interest in the study of cyclical fluctuations in macroeconomics, since periodic solutions and limit cycles resemble idealised business cycles, see for example Grandmont (1985) and a recent survey by Guesnerie and Woodford (1992). In this chapter we will analyse cycles when we have heterogeneous agents in the non-linear model from chapter 5, but we restrict ourselves to a 2-cycle to keep the dynamics simple. We will study the existence of 2-cycles together with the conditions for local stability of a 2-cycle under learning. The issue of interest is how the existence and the stability conditions of a 2-cycle under learning are affected by the introduction of heterogeneous agents in the model.

There are different ways of studying the stability of cycles under learning. Evans and Honkapohja (1995a) formulates a stochastic recursive algorithm for a stochastic non-linear model. They use a connection between the recursive algorithm and a corresponding ordinary differential equation. This connection is based on paper by Ljung (1977), and the stability of the recursive algorithm can be determined by studying the stability of the corresponding differential equation. Guesnerie and Woodford (1991) study a deterministic model with a constant speed of learning. They use the Poincare-Hopf theorem to study local asymptotic stability under learning. A paper by Grandmont and Laroque (1986) also concentrate on a non-stochastic framework, but their learning rule has finite memory and is time-invariant.

In this chapter we will adapt the framework of Evans and Honkapohja, but we keep the deterministic framework from chapter 5. The learning rules are modified in a way which allow the agents to incorporate the possibility of a 2-cycle in their beliefs. We establish a connection between the dynamic system of difference equations and a corresponding differential equation in this deterministic setting. This is in many respects similar to the recursive algorithm formulated by Ljung (1977). However the algorithm presented here is simpler, and gives an intuition on how Ljungs algorithm work.

We find a condition that secures local stability of the 2-cycle with heterogeneous learning and this does not depend on the speed of learning. This is however a sufficient condition and not a necessary condition. The result corresponds to the local stability of a 2-cycle when we have homogeneous agents. Thus if a 2-cycle is locally stable under learning in the homogeneous case it will also be locally stable if we have heterogeneous agents and vice versa. The intuition for this equivalence is that the two types of agents make the same forecast when the economy is in a 2-cycle. Hence there is no difference in their forecast and they are identical, although their speeds of learning are different.

The chapter is organised as follows. In section 6.2 we briefly describe the model and consider the conditions for existence of a 2-cycle. Section 6.3 describes the recursive algorithm we use in section 6.4 to find conditions for local stability. In section 6.5 we look at the case with homogeneous agents and show an equivalence between stability in the homogeneous case and stability in the heterogeneous case. Section 6.6 contains the conclusion.

### 6.2. The model.

We are considering the case where the dynamics are given by the function H(.,.,.):

$$p_t = H(p^e_{t+1}, q^e_{t+1}, \mu) \quad \text{for all t.}$$

This is the temporary equilibrium condition from chapter 5, where $p_t$ is the price at time t and it depends on the fast learner's expected price at time t + 1, $p^e_{t+1}$, and the slow learner's expected price at time t + 1, $q^e_{t+1}$, $\mu$ denote the fraction of fast learners. The function H was implicitly derived from the first-order conditions of the maximisation problems of the two agents in chap. 5:

$$n^f_t = \xi(p_t / p^e_{t+1}) \quad \text{and} \quad n^s_t = \xi(p_t / q^e_{t+1})$$

and the equilibrium condition on the money market:

$$M = \mu M_t^f + (1-\mu) M_t^s ,$$

where $n_t^f, M_t^f$ is the fast learners labour supply and demand for money, $M_t^s, n_t^s$ is the slow learners money demand and labour supply and M is the money supply, which is kept constant. We have :

$$M_t^f = p_t n_t^f \text{ and } M_t^s = p_t n_t^s ,$$

for all t. If we combine the first order conditions with the money market equilibrium condition, we can define the price today implicitly as a function of the expected prices:

$$M/p_t = \mu n_t^f + (1-\mu) n_t^s = \mu\xi(p_t / p_{t+1}^e) + (1-\mu)\xi(p_t / q_{t+1}^e)$$

$$p_t = H(p_{t+1}^e, q_{t+1}^e, \mu) \qquad \text{for all t.}$$

**Assumption 6.1.** H is a $C^2$-function for $p_{t+1}^e > 0$ and $q_{t+1}^e > 0$.

In chapter 5, we made an assumption on the function $\xi$ to prove that the partial derivatives of H with respect to $p_{t+1}^e$ and $q_{t+1}^e$ was positive for all values of $p_{t+1}^e > 0, q_{t+1}^e > 0$ and $0 < \mu < 1$, see assumption 5.1. This assumption, $\xi' > 0$, was sufficient to rule out cycles, hence in order to study cycles we have to drop this assumption. We will limit the case to a 2-cycle so the results should not be taken literally. The purpose of studying a 2-cycle is to simplify the set-up and to gain some intuition about the dynamics. The definition of a rational 2-cycle is as follows.

**Definition 6.1.** A rational 2-cycle is a price-pair $(\overline{p}_1, \overline{p}_2)$ where:

$$\overline{p}_1 = H(\overline{p}_2, \overline{p}_2, \mu).$$

$$\overline{p}_2 = H(\overline{p}_1, \overline{p}_1, \mu).$$

If the economy is in an 2-cycle $(\overline{p}_1, \overline{p}_2)$ then the actual law of motion is given by :

$$p_t = H(\overline{p}_1, \overline{p}_1, \mu) \qquad \text{t even.}$$

$$p_t = H(\overline{p}_2, \overline{p}_2, \mu) \qquad \text{t odd.}$$

We can write down the equilibrium conditions for this 2-cycle with the use of the equilibrium condition on the money market and the first order conditions :

$$\frac{M}{\overline{p}_1} = \mu\xi\left(\frac{\overline{p}_1}{\overline{p}_2}\right) + (1-\mu)\xi\left(\frac{\overline{p}_1}{\overline{p}_2}\right) = \xi\left(\frac{\overline{p}_1}{\overline{p}_2}\right).$$

$$\frac{M}{\overline{p}_2} = \mu\xi\left(\frac{\overline{p}_2}{\overline{p}_1}\right) + (1-\mu)\xi\left(\frac{\overline{p}_2}{\overline{p}_1}\right) = \xi\left(\frac{\overline{p}_2}{\overline{p}_1}\right).$$

Let $\overline{p}_1/\overline{p}_2 = x$ so $\overline{p}_2/\overline{p}_1 = 1/x$. The equilibrium conditions can be written as one equation :

(6.1) $\qquad G(x) = x\xi(x) - \xi(1/x) = 0.$

The steady state $\overline{p}$ is given by $x = 1$. The equilibrium condition (6.1) does not depend on the fraction of fast learners $\mu$ or slow learners $1 - \mu$. This is not surprising, since both types of agents solve the same type of maximisation problem here. In Evans et al. (1993) there is a necessary condition for the existence of a 2-cycle. If we translate their result to this model, the condition is:

$$\xi'(\overline{p}/\overline{p}) = \xi'(1) < 0.$$

Hence $\xi$ has to have a negative derivative at the steady state $\overline{p}$ for the existence of a 2-cycle.

Grandmont (1985) shows that instability of the steady state is a sufficient condition for the existence of a 2-cycle for the model with homogeneous agents. In our model, this is equivalent to :

$$H_1(\overline{p}, \overline{p}, \mu) < -\mu \text{ and } H_2(\overline{p}, \overline{p}, \mu) < -(1-\mu).$$

Although Grandmont's results are formulated for a function of one variable, H is reduced to a function of one variable when $p^e_{t+1} = q^e_{t+1} = \overline{p}$. We can then use his arguments on H to determine the conditions for existence, since the partial derivatives of H in the steady state are identical apart from $\mu$ :

$$H_1(\overline{p}, \overline{p}, \mu) = \mu\chi(\overline{p}) \text{ and } H_2(\overline{p}, \overline{p}, \mu) = (1-\mu)\chi(\overline{p}),$$

where $\chi(\overline{p}) = F'(\overline{p})$ and $F(p_{t+1}) = H(p_{t+1}, p_{t+1}, \mu)$ as defined in chapter 5. When the economy are in steady state the two types of agents have identical forecasts and they can be regarded as one type, and the condition for existence of 2-cycle in Grandmont can be translated to our model as $\chi(\bar{p}) < -1$.

**Example.** *U and V are the C.E.S.-functions from chapter 5.* Let $U(c) = c^{(1-\sigma)}/(1-\sigma)$ and $V(n) = n^{1+\varepsilon}/(1+\varepsilon)$ such that H is given by

$$H(p,q,\mu) = \{ (1/M)( \mu(p^{-\alpha}) + (1-\mu)(q^{-\alpha}) ) \}^{-\beta}$$

where $\alpha = (1-\sigma)/(\sigma+\varepsilon)$ and $\beta = (\sigma+\varepsilon)/(1+\varepsilon)$. If we set $\sigma = 3$, $\varepsilon = 1$ and $M = 1$, then the steady state is given by $\bar{p} = 1$ and there exist a 2-cycle given by $\bar{p}_1 = 1/2$ and $\bar{p}_2 = 2$.

Let us therefore assume that H satisfies these conditions so that a rational 2-cycle exists. The purpose of this chapter is to find conditions for local stability of the 2-cycle under learning. The way to study learning and the stability of cycles or steady states under learning in a stochastic setting has been to use a stochastic recursive algorithm formulated in paper by Ljung (1977). See e.g. Marcet and Sargent (1988), (1989), Woodford (1990) and Evans and Honkapohja (1995a) for applications of Ljungs algorithm. However the model in this chapter is deterministic and much simpler than the model in Ljung. Hence Ljungs algorithm might seem to powerful a tool to use. Instead we analyse a deterministic version of Ljungs algorithm in order to gain some intuition on Ljungs results and the way these results are proven.

### 6.3. A deterministic version of a stochastic recursive algorithm.

The original recursive algorithm is formulated as follows:

$$(6.2) \qquad \begin{aligned} x_t &= x_{t-1} + \gamma_t Q(t, x_{t-1}, z_t) \\ z_t &= M(x_{t-1}) \cdot z_{t-1} + N(x_{t-1}) \cdot v_t \end{aligned}$$

where $x_t \in R^n$ are the "estimates", $z_t \in R^m$ are the "observations", $v_t \in R^r$ is a random vector and $\gamma_t \in R$ is a deterministic sequence, where $\gamma_t \to 0$. M and N are $m \times m$ and $m \times r$ matrix-functions of $x_{t-1}$, respectively. In the deterministic case, $v_t \in R^r$ is a deterministic vector. Ljung shows that under certain conditions *local* convergence of

$x_t$ to a point $\bar{x}$ under (6.2) depends on the *asymptotic* stability of the ordinary differential equation:

$$\text{(6.3)} \qquad \frac{dx^D(s)}{ds} = f(x^D(s))$$

where f(x) is defined as:

$$f(x) = \lim_{t\to\infty} E(Q(t,x,\bar{z}_t))$$

and $\bar{z}_t = M(x)\bar{z}_{t-1} + N(x)v_t$, for a fixed x. The reason for using $x^D$ instead of x is to distinguish between the solution to (6.2) and the solution (6.3). The point $\bar{x}$ could, for example, be a stationary point for the differential equation, i.e. $f(\bar{x}) = 0$. Hence instead of studying the asymptotic behaviour of the stochastic difference equations system (6.2), Ljung shows under certain assumptions that we can study the asymptotic behaviour of the differential equation (6.3), see theorem 1 and 4 in Ljung.

In this chapter we have restricted ourselves to a purely deterministic model with no random variables. Let us set the matrix functions M and N equal to 0, and there no random vector $v_t$, furthermore Q does not depend on t, hence (6.2) reduces to

$$x_t = x_{t-1} + \gamma_t Q(x_{t-1}).$$

To simplify the model further we assume that $\gamma_t = 1/t$. Hence we have

$$\text{(6.2*)} \qquad x_t = x_{t-1} + \frac{1}{t} Q(x_{t-1}).$$

The system of difference equations in (6.2*) will describe our model with learning rule as presented below in section 6.4. The corresponding differential equation to (6.3) would be

$$\text{(6.3*)} \qquad \frac{dx^D(s)}{ds} = Q(x^D(s))$$

Again we distinguish between a solution to (6.3*) denoted by $x^D$ and a solution to (6.2*) denoted by $x_t$. We assume that Q is a continuous function. If we have the initial condition $x^D(s(0)) = x_0^D$, then we can write (6.3*) as

$$x^D(s(t)) = x_0^D + \int_{s(0)}^{s(t)} Q(x(s))ds \qquad \Rightarrow$$

(6.4) $$x^D(s(t)) - x_0^D = \int_{s(0)}^{s(t)} Q(x(s))ds$$

We have to find the connection between the time-index t and notional time s(t). Let us return to the difference equation at time t :

$$x_{t+1} - x_t = \frac{1}{t+1} Q(x_t) \qquad \Rightarrow$$

$$x_{t+s} - x_t = \sum_{j=t+1}^{t+s} \frac{1}{j} Q(x_{j-1}) \qquad \Rightarrow$$

(6.5) $$x_t - x_T = \sum_{j=T+1}^{t} \frac{1}{j} Q(x_{j-1})$$

The connection between t and s is given by :

$$s(t) = \sum_{j=1}^{t} \frac{1}{j}$$

We want to approximate the right-hand side of (6.5) to the integral in (6.4). Here we will use the fact that an integral can be defined as the limit of sums.

*The definition of the integral as sum of limits, see theorem 6.4 in Rudin (1964).* Let P be a partition of the interval [a,b], $a = s_0 < s_1 < ... < s_n = b$. Let $t_i$ be given by $s_{i-1} \le t_i \le s_i$. Let f be a bounded function on the interval [a,b], and the function $\alpha$ is a monotone increasing function in s with $\Delta\alpha_i = \alpha(s_i) - \alpha(s_{i-1})$. Let $\mu(P) = \max_{i=1,...,n} |s_i - s_{i-1}|$.

If f is continuous then :

$$\lim_{\mu(P)\to 0} \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \quad = \quad \int_a^b f(s)d\alpha(s)$$

This does depend upon the partition P or on the points $t_i$ as noted by Rudin. It is often assumed that $\alpha(x) = x$. □

In order to use the result, we set a = s(T) and b = s(t), where x = s and the partition P is given by {s(T), s(T+1), ...., s(t–1), s(t)} where

$$s(j) = = \sum_{i=1}^{j} \frac{1}{i} \qquad \text{for } j = T, \ldots\ldots, t.$$

$$\alpha(s_j) = s(j)$$

$$\Delta\alpha_j = \alpha(s_j) - \alpha(s_{j-1}) = s(j) - s(j-1) = 1/j$$

$$\mu(P) = \max_{i=T+1,\dots,t} (s(j) - s(j-1)) = 1/T+1.$$

$\Delta\alpha_j$ shows how fine the approximation is. The function f in the definition is equal to Q(x), such that f(s) = Q(x(s)). When T goes to infinity then $\Delta\alpha_j$ as well as $\mu(P)$ goes towards 0. Hence if we choose T sufficiently large and set $x_T = x^D(s(T))$ then for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| \sum_{j=T+1}^{t} \frac{1}{j} Q(x_{j-1}) - \int_{s(T)}^{s(t)} Q(x(s))ds \right| < \varepsilon$$

for every partition P of [s(T), s(t)] or a partition $P_t$ of [T, t] in time-t notation with $\mu(P) < \delta = 1/T$ and every choice of points in [s(j−1), s(j)], or in time-t notation [(j − 1), j], where we choose the lower bound j −1. If we choose T sufficiently large and set $x_T = x^D(s(T))$ as the initial condition, then the path of the difference equation, $x_t$, will be arbitrarily close to the path of the ordinary differential equation, $x^D(s(t))$, for t > T.

Let us assume that the differential equation (6.3*) has a stationary point , $\bar{x}$, i.e. a point where $Q(\bar{x}) = 0$. Furthermore, we assume that the real parts of the eigenvalues of the Jacobian matrix for Q at the stationary point are negative. In this case, the stationary point $\bar{x}$ is asymptotic stable as shown Brock and Mallearis (1989). Let $D_A$ be the domain of attraction for $\bar{x}$, such that any trajectory $x^D$ that starts in $D_A$ will converge to $\bar{x}$ when s goes to infinity :

$$x^D(0) \in D_A \Rightarrow x^D(s) \to \bar{x} \text{ when } s \to \infty.$$

This is illustrated in figure 6.1, where we assume that $D_A$ contains a neighbourhood W of $\bar{x}$, hence $\{ \bar{x} \}$ is an invariant set.

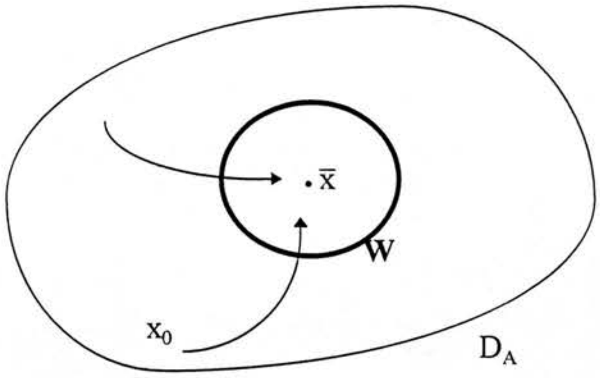



**Figure 6.1.** The stationary point and the domain of attraction $D_A$.

For a sufficiently large T the trajectories of difference equation will be approximate closely to the trajectories of differential equation for all $t > T$. If the sequence for difference equation $\{ x_t \}_t$ belongs to the neighbourhood W sufficiently often, then $x_t$ will converge to $\bar{x}$. However for small T, $x_t$ might not belong to the domain of attraction, and we must find a method to keep the $x_t$ in the domain of attraction $D_A$ until t is sufficiently large. This is also discussed briefly in Evans and Honkapohja (1995a), where they mention that is in general not possible to establish local convergence without the projection facility, because $x_t$ at an early stage might be outside the domain of attraction $D_A$. This is due to the their model being stochastic. In order to keep $x_t$ within the domain of attraction we use a projection facility defined in Ljungs paper . The projection facility is as follows : Let $D_2$ be a nontrivial, compact set containing $\bar{x}$ and let $D_1$ be an open set, where $D_2 \subset D_1$, see figure 6.2.

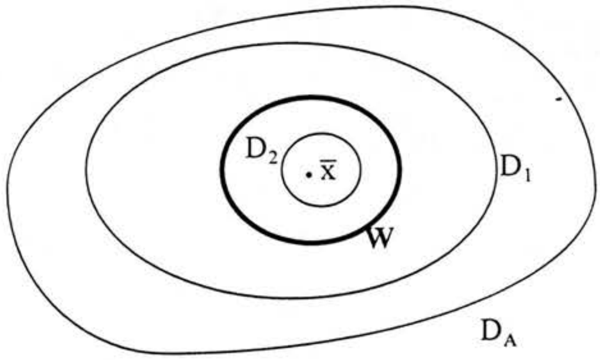



**Figure 6.2.** The projection facility.

The algorithm is then defined as:

$$x_t = \left[ x_{t-1} + \frac{1}{t} Q(x_{t-1}) \right]_{D_1, D_2} \tag{6.6}$$

Where

$$[\lambda] = \begin{cases} \lambda & \text{if } \lambda \in D_1 \\ \text{some value in } D_2 & \text{if } \lambda \notin D_1 \end{cases}$$

Where $D_1 \subset D_A$ and $D_2 \subset W$, and the initial condition for $x_t$ is given by : $x_0 \in D_2$. If for some small t, $x_t$ leaves the $D_1$ , and perhaps the domain of attraction, then the projection algorithm will project $x_t$ back into $D_2$ which is a subset of W. When t is sufficiently large, $x_t$ will still be inside $D_A$ and follow the path of the differential condition such that $x_t$ converge to $\bar{x}$. In Evans and Honkapohja (1995a) they mention that is not necessary to use the projection facility if the shock is small. They prove this with the help of a Lyapounov function. The intuition is that it is possible to show that the recursive algorithm only leave the domain of attraction a finite number of times, with the help of the Lyapounov function. Thus it will infinitely often stay inside the domain of attraction and we do not need the projection facility.

### 6.4. The local stability of the learning rule.

In the previous chapters the speed of learning was constant, this is also referred to as constant gain parameter, see Evans and Honkapohja (1993b). The agents use a similar learning rule in this chapter, but take account of the possibility that cycles of period 2 can occur as well as steady states. Furthermore, we assume that the speed of learning is no longer constant but decreasing over time. The reason for introducing the decreasing gain parameter is to be able to extend the model to cover the stochastic case, i.e. when there is a shock to the model. As noted in chapter 4, it was necessary to have a decreasing gain parameter when there were a shock in the model.

Let us assume there exist a rational 2-cycle ($\bar{p}_1, \bar{p}_2$), and the agents believe that the economy is in a rational 2-cycle with unknown parameter values. Since the two types of agents act according to the belief that the economy is a 2-cycle, the forecast of the price at time t + 1, $p_{t+1}$, should be based on data where $(\tau + 1) \bmod 2 = 0$ for $\tau \leq t$ with the following connection between t and $\tau$ :

$$t = 2\tau + i \quad \text{for } i = 1,2 \text{ and } \tau = 0,1,2,3...$$

The learning rules for the two types of agents are given by

$$x_{i,\tau} = x_{i,\tau-1} + a^f_\tau (p_{i,\tau} - x_{i,\tau-1}) \qquad \text{for } i = 1,2 \text{ and } \tau = 1,2,...$$

$$y_{i,\tau} = y_{i,\tau-1} + a^s_\tau (p_{i,\tau} - y_{i,\tau-1}) \qquad \text{for } i = 1,2 \text{ and } \tau = 1,2,...$$

where $x_{i,\tau} = p^e_{2\tau+i}$ denotes the fast learner's forecast and $y_{i,\tau} = q^e_{2\tau+i}$ denotes the slow learner's forecast at time $2\tau + i$ for i = 1,2 and $\tau$ = 0,1,2,.... The actual price is given by $p_{i,\tau} = p_{2(\tau-1)+i}$ for i = 1,2 and $\tau$ = 0,1,2... We divide the time t-axis into intervals of length 2 :

$$\overbrace{t=1 \qquad t=2}^{\tau=0} \quad \overbrace{t=3 \qquad t=4}^{\tau=1} \quad \overbrace{t=5 \qquad t=6}^{\tau=2} \ldots\ldots\ldots\ldots \overbrace{t-1 \qquad t}^{\tau} \ldots\ldots$$

This can be illustrated as follows:

The actual prices and the expected prices.

$$\left.\begin{array}{l} p_{1,1} = p_1 \\ p_{2,1} = p_2 \end{array}\right\} \tau = 1 \qquad \left.\begin{array}{l} x_{1,0} = p^e_1 \text{ and } y_{1,0} = q^e_1 \\ x_{2,0} = p^e_2 \text{ and } y_{1,0} = q^e_1 \end{array}\right\} \tau = 0.$$

$$\left.\begin{array}{l} p_{1,2}=p_3 \\ p_{2,2}=p_4 \end{array}\right\} \tau=2 \qquad \left.\begin{array}{lll} x_{1,1}=p_3^e & \text{and} & y_{1,1}=q_3^e \\ x_{2,1}=p_4^e & \text{and} & y_{1,2}=q_4^e \end{array}\right\} \tau=1.$$

$$\left.\begin{array}{l} p_{1,3}=p_5 \\ p_{2,3}=p_6 \end{array}\right\} \tau=3 \qquad \left.\begin{array}{lll} x_{1,2}=p_5^e & \text{and} & y_{1,2}=q_5^e \\ x_{2,2}=p_6^e & \text{and} & y_{2,2}=q_6^e \end{array}\right\} \tau=2.$$

and so forth.

Hence the fast learners update their forecast of an odd time price based on the previous odd time forecast and the previous odd time actual price. The forecast of an even time price is based at the previous even time forecast and actual price. The slow learners use a similar method but have a different speed of learning.

The learning rules can be rearranged into 4-tuplets such that we have the algorithm :

$$\text{(6.7)} \qquad \begin{pmatrix} x_{1,\tau} \\ x_{2,\tau} \\ y_{1,\tau} \\ y_{2,\tau} \end{pmatrix} = \left[ \begin{pmatrix} x_{1,\tau-1} \\ x_{2,\tau-1} \\ y_{1,\tau-1} \\ y_{2,\tau-1} \end{pmatrix} + \begin{pmatrix} a_\tau^f(p_{1,\tau}-x_{1,\tau-1}) \\ a_\tau^f(p_{2,\tau}-x_{2,\tau-1}) \\ a_\tau^s(p_{1,\tau}-y_{1,\tau-1}) \\ a_\tau^s(p_{2,\tau}-y_{2,\tau-1}) \end{pmatrix} \right]_{D_1,D_2}$$

where $D_1$, $D_2$ denotes the projection facility from section 6.3. The connection between actual price and expected prices is given by the function H:

$$p_t = H(p_{t+1}^e, q_{t+1}^e)$$

where $\mu$ has been omitted for simplicity. Hence we have

$$p_{i,\tau} = p_{2(\tau-1)+i} = p_{2\tau-1} = H(p_{2\tau}^e, q_{2\tau}^e) \qquad \text{if } i=1,$$

such that the forecast at an even time, $(p_{2\tau}^e, q_{2\tau}^e)$, is used to calculate the actual price at an odd time, $p_{1,\tau} = p_{2\tau-1}$ and

$$p_{1,\tau} = H(x_{2,(\tau-1)}, y_{2,(\tau-1)}) \qquad \text{if } i=1.$$

The actual price at an even time, $p_{2\tau}$, is calculated by the forecast at an odd time, $(p_{2\tau+1}^e, q_{2\tau+1}^e)$:

$$p_{i,\tau} = p_{2(\tau-1)+i} = p_{2\tau} = H(p_{2\tau+1}^e, q_{2\tau+1}^e) \qquad \text{if } i=2,$$

or $\qquad p_{2,\tau} = H(x_{1,\tau}, y_{1,\tau})$

This can be is inserted into the algorithm (6.7):

(6.8) $$\begin{pmatrix} x_{1,\tau} \\ x_{2,\tau} \\ y_{1,\tau} \\ y_{2,\tau} \end{pmatrix} = \left[ \begin{pmatrix} x_{1,\tau-1} \\ x_{2,\tau-1} \\ y_{1,\tau-1} \\ y_{2,\tau-1} \end{pmatrix} + \begin{pmatrix} a^{f}_{\tau}(H(x_{2,\tau-1}, y_{2,\tau-1}) - x_{1,\tau-1}) \\ a^{f}_{\tau}(H(x_{1,\tau}, y_{1,\tau}) - x_{2,\tau-1}) \\ a^{s}_{\tau}(H(x_{2,\tau-1}, y_{2,\tau-1}) - y_{1,\tau-1}) \\ a^{s}_{\tau}(H(x_{1,\tau}, y_{1,\tau}) - y_{2,\tau-1}) \end{pmatrix} \right]_{D_1,D_2}$$

As can be seen from row 2 and 4, the algorithm (6.8) is not fully recursive. Hence we substitute $x_{1,\tau}$ and $y_{1,\tau}$ with $x_{1,\tau} = x_{1,\tau-1} + a^{f}_{\tau}(p_{1,\tau} - x_{1,\tau-1})$ and $y_{1,\tau} = y_{1,\tau-1} + a^{s}_{\tau}(p_{1,\tau} - y_{1,\tau-1})$ into $H(x_{1,\tau}, y_{1,\tau})$ of row 2 and 4 of (6.8) :

(6.9) $$\begin{pmatrix} x_{1,\tau} \\ x_{2,\tau} \\ y_{1,\tau} \\ y_{2,\tau} \end{pmatrix} = \left[ \begin{pmatrix} x_{1,\tau-1} \\ x_{2,\tau-1} \\ y_{1,\tau-1} \\ y_{2,\tau-1} \end{pmatrix} + \begin{pmatrix} a^{f}_{\tau}(H(x_{2,\tau-1}, y_{2,\tau-1}) - x_{1,\tau-1}) \\ a^{f}_{\tau}(H(\psi(x_{1,\tau-1}), \psi(y_{1,\tau-1})) - x_{2,\tau-1}) \\ a^{s}_{\tau}(H(x_{2,\tau-1}, y_{2,\tau-1}) - y_{1,\tau-1}) \\ a^{s}_{\tau}(H(\psi(x_{1,\tau-1}), \psi(y_{1,\tau-1})) - y_{2,\tau-1}) \end{pmatrix} \right]_{D_1,D_2}$$

where $\psi(x_{1,\tau-1}) = x_{1,\tau-1} + a^{f}_{\tau}(p_{1,\tau} - x_{1,\tau-1})$ and $\psi(y_{1,\tau-1}) = y_{1,\tau-1} + a^{s}_{\tau}(p_{1,\tau} - y_{1,\tau-1})$, with $p_{1,\tau} = H(x_{2,\tau-1}, y_{2,\tau-1})$. Now the algorithm is fully recursive. In the recursive algorithm from section 6.3, it was necessary to have the same $\gamma_\tau$ in front of all the rows in the matrix in (6.9), hence the following assumption is made on the speed of learning, and as mentioned above we furthermore assume the speed of learning is decreasing. This is captured by assumption 6.2.

**Assumption 6.2.** $a^{f}_{\tau} = \gamma_\tau a_f$ and $a^{s}_{\tau} = \gamma_\tau a_s$ for all $\tau \geq 1$, where $\gamma_\tau = \frac{1}{\tau}$ for all $\tau$.

We have a decreasing gain parameter in the algorithm and the algorithm can be written as:

(6.10) $$\begin{pmatrix} x_{1,\tau} \\ x_{2,\tau} \\ y_{1,\tau} \\ y_{2,\tau} \end{pmatrix} = \left[ \begin{pmatrix} x_{1,\tau-1} \\ x_{2,\tau-1} \\ y_{1,\tau-1} \\ y_{2,\tau-1} \end{pmatrix} + \frac{1}{\tau} \begin{pmatrix} a_f(H(x_{2,\tau-1}, y_{2,\tau-1}) - x_{1,\tau-1}) \\ a_f\,(H(\psi(x_{1,\tau-1}), \psi(y_{1,\tau-1}) - x_{2,\tau-1}) \\ a_s(H(x_{2,\tau-1}, y_{2,\tau-1}) - y_{1,\tau-1}) \\ a_s(H(\psi(x_{1,\tau-1}), \psi(y_{1,\tau-1})) - y_{2,\tau-1}) \end{pmatrix} \right]_{D_1,D_2}$$

for $\tau = 1,2,3$...or in vector notation:

(6.11) $$X_\tau = [\ X_{\tau-1} + \frac{1}{\tau} Q(X_{\tau-1})\ ]_{D_1,D_2} \qquad \text{for } \tau = 1,2,3.....$$

where $X_\tau$ denote the vector $(x_{1,\tau}, x_{2,\tau}, y_{1,\tau}, y_{2,\tau})$, and for example the first row and third row in the Q-matrix is given by:

$$Q_1(X_{\tau-1}) = a_f(H(x_{2,\tau-1}, y_{2,\tau-1}) - x_{1,\tau-1}) \qquad \text{and}$$
$$Q_3(X_{\tau-1}) = a_s(H(\psi(x_{2,\tau-1}), \psi(y_{2,\tau-1})) - y_{1,\tau-1}).$$

We can now use the results from section 6.3 on this algorithm. We furthermore assume that $\gamma_\tau = 1/\tau$ for all $\tau$.

The local stability of (6.11) is given by the local asymptotic stability of the associated differential equation according to the results in section 6.3.

$$\text{(6.12)} \qquad \frac{dX^D(r)}{dr} = Q(X^D(r)).$$

where $r \geq 0$ is notional time. The purpose is to study whether $X_\tau = (x_{1,\tau}, x_{2,\tau}, y_{1,\tau}, y_{2,\tau})$ converge to $\overline{X} = (\overline{p}_1, \overline{p}_2, \overline{p}_1, \overline{p}_2)$ when $\tau$ goes to infinity for initial values $X_1$ close to $\overline{X}$, i.e. $X_1 \in W$ where W is a neighbourhood around $\overline{X}$ as defined in section 6.3, and $\overline{X}$ is a stationary point for Q.

In order to determine local asymptotic stability of $\overline{X}$ under (6.12), we have to find the eigenvalues of the Jacobian matrix for $Q(X(r))$ at the stationary point $\overline{X}$. If all the eigenvalues of the Jacobian have a negative real part, then the differential equation is locally stable at the stationary point $\overline{X}$, and the algorithm converges to $\overline{X}$ for nearby starting points. Let us therefore consider the Jacobian for Q at $\overline{X}$. This is a 4×4-matrix given by:

$$J(\overline{X}) = \begin{pmatrix} -a_f & a_f H_1(\overline{p}_2, \overline{p}_2) & 0 & a_f H_2(\overline{p}_2, \overline{p}_2) \\ a_f H_1(\overline{p}_1, \overline{p}_1) & -a_f & a_f H_2(\overline{p}_1, \overline{p}_1) & 0 \\ 0 & a_s H_1(\overline{p}_2, \overline{p}_2) & -a_s & a_s H_2(\overline{p}_2, \overline{p}_2) \\ a_s H_1(\overline{p}_1, \overline{p}_1) & 0 & a_s H_2(\overline{p}_1, \overline{p}_1) & -a_s \end{pmatrix}$$

where $H_1$ and $H_2$ are the partial derivatives of H w.r.t. $p_t^e$ and $q_t^e$, respectively. From chapter 5, we have the following property of the partial derivatives for H at $(\overline{p}_1, \overline{p}_1)$ and $(\overline{p}_2, \overline{p}_2)$:

$$H_1(\overline{p}_1, \overline{p}_1) = \mu\chi(\overline{p}_1) \text{ and } H_2(\overline{p}_1, \overline{p}_1) = (1-\mu)\chi(\overline{p}_1)$$

$$H_2(\bar{p}_2, \bar{p}_2) = \mu\chi(\bar{p}_2) \text{ and } H_2(\bar{p}_2, \bar{p}_2) = (1-\mu)\chi(\bar{p}_2),$$

where $\chi(\bar{p}_1) = F'(\bar{p}_1)$ and $\chi(\bar{p}_2) = F'(\bar{p}_2)$. We can find the eigenvalues of $J(\bar{X})$, but the expressions are not very nice:

$$\lambda_1 = -\frac{1}{2}(a_f + a_s) - \frac{1}{2}\sqrt{(i)} - \frac{1}{2}\sqrt{(ii) - \frac{(iii)}{4\cdot\sqrt{(i)}}}$$

$$\lambda_2 = -\frac{1}{2}(a_f + a_s) - \frac{1}{2}\sqrt{(i)} + \frac{1}{2}\sqrt{(ii) - \frac{(iii)}{4\cdot\sqrt{(i)}}}$$

$$\lambda_3 = -\frac{1}{2}(a_f + a_s) + \frac{1}{2}\sqrt{(i)} - \frac{1}{2}\sqrt{(ii) + \frac{(iii)}{4\cdot\sqrt{(i)}}}$$

$$\lambda_4 = -\frac{1}{2}(a_f + a_s) + \frac{1}{2}\sqrt{(i)} + \frac{1}{2}\sqrt{(ii) + \frac{(iii)}{4\cdot\sqrt{(i)}}}$$

with $(i) = ((1-\mu)a_s + \mu a_f)^2 \chi(\bar{p}_1)\chi(\bar{p}_2)$,

$(ii) = (a_f - a_s)^2 + ((1-\mu)\, a_s + \mu a_f)^2 \chi(\bar{p}_1)\chi(\bar{p}_2))$ and

$(iii) = 8(a_f - a_s)\, \chi(\bar{p}_1)\chi(\bar{p}_2)\{((1-\mu)a_s + \mu a_f)((1-\mu)a_s - \mu a_f)\}$.

If $\chi(\bar{p}_1)\chi(\bar{p}_2) < 0$ then the real part of all the eigenvalues are negative, since the real part is equal to $-1/2(a_f + a_s)$ for all four eigenvalues. When $\chi(\bar{p}_1)\chi(\bar{p}_2) = 0$ then the eigenvalues are equal to either $-a_f$ or $-a_s$ and all the eigenvalues are negative. If $\chi(\bar{p}_1)\chi(\bar{p}_2) > 0$, then it is uncertain whether the eigenvalues have a negative real part and we have to check the size of the eigenvalues. In the appendix we show that all four eigenvalues are negative if $F'(\bar{p}_1)F'(\bar{p}_2) = \chi(\bar{p}_1)\chi(\bar{p}_2) < 1$. We can summarise the result.

**Proposition 6.1.** *Given assumption 6.1 and 6.2, if $\chi(\bar{p}_1)\chi(\bar{p}_2) < 1$ then the 2-cycle $(\bar{p}_1, \bar{p}_2)$ is locally stable under learning.*

**Proof.** See appendix. ■

We should note that the condition in proposition 6.1 is a sufficient condition and not necessary condition.

**Example continued.** When $\varepsilon = 1$, $\sigma = 3$ and $M = 1$, then

$$\chi(\bar{p}_1) = \alpha\beta(\bar{p}_1)^{\alpha\beta-1} = -1$$

$$\chi(\bar{p}_2) = \alpha\beta(\bar{p}_2)^{\alpha\beta-1} = -1$$

and $\chi(\bar{p}_1)\chi(\bar{p}_2) = 1$ in this special case and if we set $\mu = 0.5$, $a_f = 0.9$ and $a_s = 0.5$, then the eigenvalues are given by

$$\lambda_1 = -1.5, \lambda_2 = -0.7, \lambda_3 = -0.6, \lambda_4 = 0,$$

and the 2-cycle $(\bar{p}_1, \bar{p}_2) = (0.5, 2)$ can be unstable. However in this case the condition in proposition 6.1 is not satisfied. If we instead choose $M = 0.5$ but keep the other parameter values then $(\bar{p}_1, \bar{p}_2) = (1/6, 3/2)$ is a 2-cycle and

$$\chi(\bar{p}_1) = (M^{\beta})\alpha\beta(\bar{p}_1)^{\alpha\beta-1} = -0.25$$

$$\chi(\bar{p}_2) = (M^{\beta})\alpha\beta(\bar{p}_2)^{\alpha\beta-1} = -0.25$$

and $\chi(\bar{p}_1)\chi(\bar{p}_2) = 1/16$. In this case the eigenvalues are

$$\lambda_1 \approx -1.03, \lambda_2 \approx -0.80, \lambda_3 \approx -0.54, \lambda_4 \approx -0.42$$

and the 2-cycle is stable. □

Let us now compare the stability condition in proposition 6.1 with the stability condition for a rational 2-cycle when we have homogeneous agents.

## 6.5. The 2-cycle and homogeneous agents.

In order to make the comparison between homogenous and heterogeneous agents, we consider the case where $\mu = 1$. The case where $\mu = 0$ is similar. When $\mu = 1$ there is only one first-order condition

$$\mu = 1: \quad n_t = \xi\left(\frac{p_t}{p^e_{t+1}}\right).$$

The equilibrium condition is reduced to :

$$\text{(6.13)} \qquad \mu = 1: \quad \frac{M}{p_t} = \xi\left(\frac{p_t}{p^e_{t+1}}\right) \;\Rightarrow\; p_t = F(p^e_{t+1}).$$

The F-function only depends on $p^e_{t+1}$ and we assume that F is a $C^2$-function in $p^e_{t+1}$.

The definition of a 2-cycle is similar to the previous definition, hence a rational 2-cycle is a pair $(\overline{p}_1, \overline{p}_2)$ where

$$\overline{p}_1 = F(\overline{p}_2) \text{ and } \overline{p}_2 = F(\overline{p}_1).$$

We can illustrate the function F and the 2-cycle :

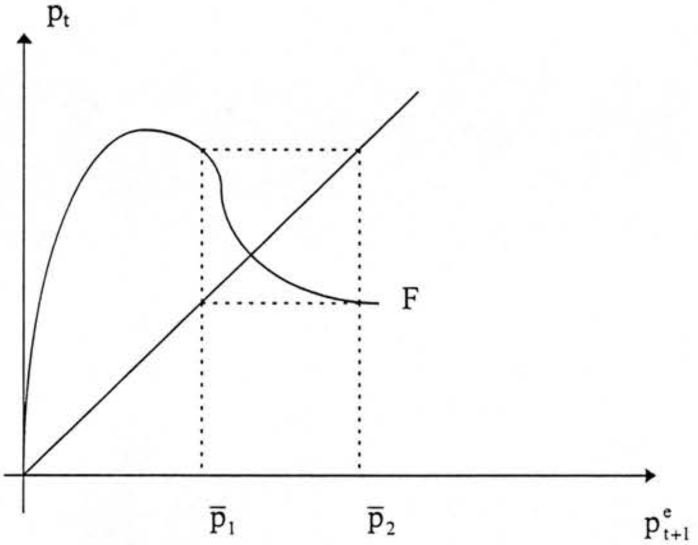



**Figure 6.1.** The F-function with $\mu = 1$.

We have to show that a pair $(\overline{p}_1, \overline{p}_2)$ is a 2-cycle in the homogeneous case if and only if it is a 2-cycle in the heterogeneous case defined in section 6.2. Assume the opposite, then there are two different pairs such that $(\widetilde{p}_1, \widetilde{p}_2)$ is a 2-cycle determined by the H-function and $(\overline{p}_1, \overline{p}_2)$ is a 2-cycle determined by the F-function. The 2-cycle $(\widetilde{p}_1, \widetilde{p}_2)$ has to satisfy the equilibrium conditions given by :

$$\frac{M}{\widetilde{p}_1} = \mu\xi(\frac{\widetilde{p}_1}{\widetilde{p}_2}) + (1-\mu)\xi(\frac{\widetilde{p}_1}{\widetilde{p}_2}) = \xi(\frac{\widetilde{p}_1}{\widetilde{p}_2}) \qquad \text{and}$$

$$\frac{M}{\widetilde{p}_2} = \mu\xi(\frac{\widetilde{p}_2}{\widetilde{p}_1}) + (1-\mu)\xi(\frac{\widetilde{p}_2}{\widetilde{p}_1}) = \xi(\frac{\widetilde{p}_2}{\widetilde{p}_1})$$

while a 2-cycle $(\overline{p}_1, \overline{p}_2)$ for F satisfies :

$$\frac{M}{\overline{p}_1} = \xi(\frac{\overline{p}_1}{\overline{p}_2}) \quad \text{and} \quad \frac{M}{\overline{p}_2} = \xi(\frac{\overline{p}_2}{\overline{p}_1})$$

Thus a 2-cycle in the heterogeneous case satisfies the equilibrium condition if and only if it satisfies the condition for a 2-cycle in the homogeneous case.

Let us, corresponding to section 6.4, assume that the agents act as if the economy were in a 2-cycle and their forecast is again based on data where $(\tau + 1) \bmod 2 = 0$ for $\tau \leq t$. Let $t = 2\tau + i$ for $i = 1,2$ and $\tau = 1,2,3...$ and let the agents make their forecast according to the adaptive learning rule from section 6.4 :

$$x_{1,\tau} = x_{1,\tau-1} + a^f_\tau(p_{1,\tau} - x_{1,\tau-1}) \qquad \tau = 1,2,...$$

$$x_{2,\tau} = x_{2,\tau-1} + a^f_\tau(p_{2,\tau} - x_{2,\tau-1}) \qquad \tau = 1,2,...$$

where $x_{i,\tau} = p^e_{2\tau+i}$ is the expected price at time $2\tau + i$ for $i = 1,2$ and $\tau = 1,2,....$ The actual price is given by $p_{i,\tau} = p_{2(\tau-1)+i}$ for $i = 1,2$ and $\tau = 1,2...$ We use the forecast at an even time to update the price in odd periods such that the connection between the actual price $p_{i,\tau}$ and the expected price when $i = 1$ is given by :

$$p_{2\tau-1} = F(p^e_{2\tau}) \qquad \text{or}$$

$$p_{1,\tau} = F(x_{2,\tau-1}) \qquad \tau = 1,2...$$

When $i = 2$, we use the forecast at an odd time to update a price at an even time:

$$p_{2\tau} = F(p^e_{2\tau+1}) \qquad \text{or}$$

$$p_{2,\tau} = F(x_{1,\tau}) \qquad \tau = 1,2,...$$

This is inserted into the learning rule such that we have the following algorithm:

$$\begin{pmatrix} x_{1,\tau} \\ x_{2,\tau} \end{pmatrix} = \begin{pmatrix} x_{1,\tau-1} \\ x_{2,\tau-1} \end{pmatrix} + \begin{pmatrix} a^f_\tau(F(x_{2,\tau-1}) - x_{1,\tau-1}) \\ a^f_\tau(F(x_{1,\tau}) - x_{2,\tau-1}) \end{pmatrix}$$

This is not fully recursive, hence we substitute $x_{1,\tau}$ with $x_{1,\tau} = x_{1,\tau-1} + a^f_\tau(p_{1,\tau} - x_{1,\tau-1})$ into $F(x_{1,\tau})$ :

$$\begin{pmatrix} x_{1,\tau} \\ x_{2,\tau} \end{pmatrix} = \left[\begin{pmatrix} x_{1,\tau-1} \\ x_{2,\tau-1} \end{pmatrix} + \begin{pmatrix} a^f_\tau(\ F(x_{2,\tau-1}) - x_{1,\tau-1}) \\ a^f_\tau(\ F(\psi(x_{1,\tau-1})) - x_{2,\tau-1}) \end{pmatrix}\right]_{D_1,D_2}$$

where $\psi(x_{1,\tau-1}) = x_{1,\tau-1} + a^f_\tau(p_{1,\tau} - x_{1,\tau-1})$ with $p_{1,\tau} = F(x_{2,\tau-1})$ so that the algorithm is fully recursive, and $D_1$, $D_2$ refers to the projection algorithm . When we had heterogeneous agents we made two assumptions on the speed of learning $a^f_\tau$. These assumptions are also made here

**Assumption 6.3.** $a_\tau^f = \gamma_\tau a_f$, where $\gamma_\tau = 1/\tau$ and $0 < a_f < 1$.

Hence the speed of learning is again decreasing, and the recursive algorithm can be written as:

$$\begin{pmatrix} x_{1,s} \\ x_{2,s} \end{pmatrix} = \left[ \begin{pmatrix} x_{1,\tau-1} \\ x_{2,\tau-1} \end{pmatrix} + a_\tau^f \begin{pmatrix} F(x_{2,\tau-1}) - x_{1,\tau-1} \\ F(\psi(x_{1,\tau-1})) - x_{2,\tau-1} \end{pmatrix} \right]_{D_1, D_2}$$

for $\tau$ = 1,2,3.. . The issue is still the local stability of the rational 2-cycle $(\bar{p}_1, \bar{p}_2)$ under the recursive algorithm. We can use the results from section 6.3 as we did in the heterogeneous case and local stability is determined by the differential equation :

$$\frac{dX^D(r)}{dr} = Q(X^D(r))$$

where Q is defined by :

$$Q(X) = \begin{pmatrix} a_f(F(x_2) - x_1) \\ a_f(F(x_1) - x_2) \end{pmatrix}$$

with $X = (x_1, x_2)$. The stability of the rational 2-cycle $(\bar{p}_1, \bar{p}_2)$ is thus determined by the eigenvalues for the Jacobian matrix :

$$J(\bar{p}_1, \bar{p}_2) = \begin{pmatrix} -a_f & a_f F'(\bar{p}_2) \\ a_f F'(\bar{p}_1) & -a_f \end{pmatrix}.$$

Since J is a 2×2-matrix, the eigenvalues have negative real parts if the trace of J is negative and the determinant of J is positive. The trace of $J(\bar{p}_1, \bar{p}_2)$ is $-2a_f$ and the determinant is

$$\det(J(\bar{p}_1, \bar{p}_2)) = (a_f)^2(1 - F'(\bar{p}_1)F'(\bar{p}_2)).$$

and local stability is ensured if

$$F'(\bar{p}_1)F'(\bar{p}_2) < 1.$$

In this case the real parts of the eigenvalues for $J(\bar{p}_1, \bar{p}_2)$ are negative. Hence a necessary and sufficient condition for stability is $F'(\bar{p}_1)F'(\bar{p}_2) < 1$.

**Proposition 6.2.** *Given assumption 6.3. If $F'(\bar{p}_1)F'(\bar{p}_2) < 1$, then the 2-cycle $(\bar{p}_1, \bar{p}_2)$ is locally stable under learning.*

We can compare this condition with the stability condition for a 2-cycle, when we have heterogeneous agents. The dynamics in the heterogeneous case is governed by the function H, which is a function of $p^e_{t+1}$ and $q^e_{t+1}$, such that the partial derivatives at $(p^e_{t+1}, q^e_{t+1}) = (\bar{p}_1, \bar{p}_1)$ or $(p^e_{t+1}, q^e_{t+1}) = (\bar{p}_2, \bar{p}_2)$ are given by :

$H_1(\bar{p}_1, \bar{p}_1, \mu) = \mu\chi(\bar{p}_1)$ and $H_2(\bar{p}_1, \bar{p}_1, \mu) = (1-\mu)\chi(\bar{p}_1)$

$H_1(\bar{p}_2, \bar{p}_2, \mu) = \mu\chi(\bar{p}_2)$ and $H_2(\bar{p}_2, \bar{p}_2, \mu) = (1-\mu)\chi(\bar{p}_2)$

However when $(p^e_{t+1}, q^e_{t+1}) = (\bar{p}_1, \bar{p}_1)$ (or $(\bar{p}_2, \bar{p}_2)$) both agents have the same forecast $\bar{p}_1$ (or $\bar{p}_2$), and can be regarded as the same type even though their speed of learning differ. In fact we will show that

$$H_1(\bar{p}_1, \bar{p}_1, \mu) = \mu F'(\bar{p}_1) \text{ and } H_1(\bar{p}_2, \bar{p}_2, \mu) = \mu F'(\bar{p}_2).$$

$$H_2(\bar{p}_1, \bar{p}_1, \mu) = (1-\mu) F'(\bar{p}_1) \text{ and } H_2(\bar{p}_2, \bar{p}_2, \mu) = (1-\mu) F'(\bar{p}_2).$$

If we look at the partial derivatives $H_1$ and $H_2$ when $\mu \neq 1$, we have from the proof of lemma 5.1 in chapter 5 :

$$\frac{\partial p_t}{\partial p^e_{t+1}} = \mu \frac{1}{\Lambda} \xi'\left(\frac{p_t}{p^e_{t+1}}\right)\left(\frac{p_t}{p^e_{t+1}}\right)^2 \text{ and } \frac{\partial p_t}{\partial q^e_{t+1}} = (1-\mu) \frac{1}{\Lambda} \xi'\left(\frac{p_t}{q^e_{t+1}}\right)\left(\frac{p_t}{q^e_{t+1}}\right)^2$$

where $\Lambda = \left(\frac{M}{p_t} + \mu\xi'\left(\frac{p_t}{p^e_{t+1}}\right)\frac{p_t}{p^e_{t+1}} + (1-\mu)\xi'\left(\frac{p_t}{q^e_{t+1}}\right)\frac{p_t}{q^e_{t+1}}\right)$. For example, $(p^e_{t+1}, q^e_{t+1}) = (\bar{p}_1, \bar{p}_1)$ such that $p_t = \bar{p}_2$ then

$$\text{(6.14a)} \qquad \frac{\partial p_t}{\partial p^e_{t+1}}(\bar{p}_1, \bar{p}_1)\bigg|_{0<\mu<1} = \mu \frac{1}{\left(\frac{M}{\bar{p}_2} + \xi'\left(\frac{\bar{p}_2}{\bar{p}_1}\right)\frac{\bar{p}_2}{\bar{p}_1}\right)} \xi'\left(\frac{\bar{p}_2}{\bar{p}_1}\right)\left(\frac{\bar{p}_2}{\bar{p}_1}\right)^2$$

$$\text{(6.14b)} \qquad \frac{\partial p_t}{\partial p^e_{t+1}}(\bar{p}_2, \bar{p}_2)\bigg|_{0<\mu<1} = \mu \frac{1}{\left(\frac{M}{\bar{p}_1} + \xi'\left(\frac{\bar{p}_1}{\bar{p}_2}\right)\frac{\bar{p}_1}{\bar{p}_2}\right)} \xi'\left(\frac{\bar{p}_1}{\bar{p}_2}\right)\left(\frac{\bar{p}_1}{\bar{p}_2}\right)^2$$

$$\text{(6.15a)} \qquad \frac{\partial p_t}{\partial q^e_{t+1}}(\bar{p}_1, \bar{p}_1)\bigg|_{0<\mu<1} = (1-\mu) \frac{1}{\left(\frac{M}{\bar{p}_2} + \xi'\left(\frac{\bar{p}_2}{\bar{p}_1}\right)\frac{\bar{p}_2}{\bar{p}_1}\right)} \xi'\left(\frac{\bar{p}_2}{\bar{p}_1}\right)\left(\frac{\bar{p}_2}{\bar{p}_1}\right)^2$$

(6.15b) $$\frac{\partial p_t}{\partial q^e_{t+1}}(\bar{p}_2,\bar{p}_2)\bigg|_{0<\mu<1} = (1-\mu)\frac{1}{\left(\frac{M}{\bar{p}_1} + \xi'\left(\frac{\bar{p}_1}{\bar{p}_2}\right)\frac{\bar{p}_1}{\bar{p}_2}\right)}\xi'\left(\frac{\bar{p}_1}{\bar{p}_2}\right)\left(\frac{\bar{p}_1}{\bar{p}_2}\right)^2.$$

If we consider the partial derivative when we have homogeneous agents, $\mu = 1$,

$$\frac{\partial p_t}{\partial p^e_{t+1}} = \frac{1}{\Lambda}\xi'\left(\frac{p_t}{p^e_{t+1}}\right)\left(\frac{p_t}{p^e_{t+1}}\right)^2$$

where $\Lambda = \dfrac{M}{p_t} + \xi'\left(\dfrac{p_t}{p^e_{t+1}}\right)\dfrac{p_t}{p^e_{t+1}}$ and with $p^e_{t+1} = \bar{p}_1$ such that $p_t = \bar{p}_2$ then

(6.16) $$\frac{\partial p_t}{\partial p^e_{t+1}}(\bar{p}_1)\bigg|_{\mu=1} = \frac{1}{\left(\frac{M}{\bar{p}_2} + \xi'(\frac{\bar{p}_2}{\bar{p}_1})\frac{\bar{p}_2}{\bar{p}_1}\right)}\xi'\left(\frac{\bar{p}_2}{\bar{p}_1}\right)\left(\frac{\bar{p}_2}{\bar{p}_1}\right)^2.$$

Therefore we can write (6.16) as a function of (6.14) :

$$\frac{\partial p_t}{\partial p^e_{t+1}}(\bar{p}_1,\bar{p}_1)\bigg|_{0<\mu<1} = \mu\frac{\partial p_t}{\partial p^e_{t+1}}(\bar{p}_1)\bigg|_{\mu=1}$$

$$\frac{\partial p_t}{\partial p^e_{t+1}}(\bar{p}_2,\bar{p}_2)\bigg|_{0<\mu<1} = \mu\frac{\partial p_t}{\partial p^e_{t+1}}(\bar{p}_2)\bigg|_{\mu=1}.$$

A similar argument applies to $\partial p_t/\partial q^e_{t+1}$ but with $(1-\mu)$ instead of $\mu$, and we have the following connection between the partial derivatives for the function H with heterogeneous agents and the derivatives for the function F with homogeneous agents

$$H_1(\bar{p}_1,\bar{p}_1,\mu) = \mu F'(\bar{p}_1) \quad \text{and} \quad H_2(\bar{p}_1,\bar{p}_1,\mu) = (1-\mu)F'(\bar{p}_1)$$

$$H_1(\bar{p}_2,\bar{p}_2,\mu) = \mu F'(\bar{p}_2) \quad \text{and} \quad H_2(\bar{p}_2,\bar{p}_2,\mu) = (1-\mu)F'(\bar{p}_2).$$

This connection is not surprising, for example. when $q^e_{t+1} = p^e_{t+1} = \bar{p}_1$, then H reduces to a function of one variable, actually H is equal to F in this situation. If the 2-cycle is locally stable under learning in the homogeneous case, then it also locally stable under learning in the heterogeneous case, since

$$F'(\bar{p}_1)F'(\bar{p}_2) < 1 \qquad \Rightarrow$$

$$\chi(\bar{p}_1)\chi(\bar{p}_2) < 1,$$

We can also do the opposite, such that stability in the heterogeneous case implies stability in the homogeneous case. We have thus established an connection between the local stability conditions for a 2-cycle under learning for homogeneous agents and heterogeneous agents.

### 6.6. Conclusion.

In this chapter we have studied 2-cycles and the local stability of a 2-cycle under heterogeneous learning. This was done in a set-up with decreasing speed of learning, but in a deterministic model. The existence of 2-cycle could be established with the help of existing results where we have homogeneous agents. We found a set of conditions for the 2-cycle to locally stable under learning, the idea was to establish a connection between a difference equation and a differential equation and study the stability of the associated differential equation. The stability conditions in the heterogeneous case corresponded to the stability conditions in the homogeneous case.

There are various extensions we could study, first it would be natural to look at a k-cycle and extend the number of learners to n. We could also look at global asymptotic stability of the cycle under learning, and investigate whether the domain of attraction for which trajectories converge to the cycle depend the size of the speed of learning. Finally we could look at stochastic algorithm, for example, if the function H depends on a productivity shock but this is left for future research.

If the cycle is unstable, this can give rise to more complex situations as noted in Guesnerie and Woodford (1991) and in more detail in Bullard (1994). If the 2-cycle does not exist and the steady state is unstable, then it is possible that the economy could converge to a strange attractor as shown in Bullard. This attractor arises as a bifurcation phenomena in Bullards model. It might seem strange that the agents can detect some periodic motion and not a 2-cycle or the steady state. The agents do not need to be too smart in order to spot a 2-cycle, but in Bullards case the assumptions the ensures the existence of a strange attractor exclude the possibility of a cycle.

## Appendix to chapter 6.

Here we will prove that the eigenvalues of $J(\overline{X})$ have a negative real part.

*Proof of proposition 6.1.* Given assumption 6.1, the eigenvalues of $J(\overline{X})$ are :

$$\lambda_1 = -\frac{1}{2}(a^f + a^\ell) - \frac{1}{2}\sqrt{(1)} - \frac{1}{2}\sqrt{(2) - \frac{(3)}{4\cdot\sqrt{(1)}}}$$

$$\lambda_2 = -\frac{1}{2}(a_f + a_s) - \frac{1}{2}\sqrt{(1)} + \frac{1}{2}\sqrt{(2) - \frac{(3)}{4\cdot\sqrt{(1)}}}$$

$$\lambda_3 = -\frac{1}{2}(a_f + a_s) + \frac{1}{2}\sqrt{(1)} - \frac{1}{2}\sqrt{(2) + \frac{(3)}{4\cdot\sqrt{(1)}}}$$

$$\lambda_4 = -\frac{1}{2}(a_f + a_s) + \frac{1}{2}\sqrt{(1)} + \frac{1}{2}\sqrt{(2) + \frac{(3)}{4\cdot\sqrt{(1)}}}$$

where $(1) = (\,(1-\mu)a_s + \mu a_f)^2\,\chi(\overline{p}_1)\chi(\overline{p}_2)$,

$(2) = (a_f - a_s)^2 + (\,(1-\mu)\,a_s + \mu a_f)^2\,\chi(\overline{p}_1)\chi(\overline{p}_2)$ and

$(3) = 8(a_f - a_s)\chi(\overline{p}_1)\chi(\overline{p}_2)\{\,(\,(1-\mu)a_s + \mu a_f)((1-\mu)a_s - \mu a_f)\,\}$.

If the real part of the four eigenvalues is negative then the 2-cycle is locally stable. We divide this into two cases either $\chi(\overline{p}_1)\chi(\overline{p}_2) \le 0$ or $\chi(\overline{p}_1)\chi(\overline{p}_2) > 0$. We have already shown that the real part of all the eigenvalues are negative if $\chi(\overline{p}_1)\chi(\overline{p}_2) \le 0$. Hence we only have to look at the case where $\chi(\overline{p}_1)\chi(\overline{p}_2) > 0$.

Let us assume that $\chi(\overline{p}_1)\chi(\overline{p}_2) > 0$. Look first at $\lambda_1$ and $\lambda_2$, if $(2) - ((3)/(4\sqrt{(1)}\,)) < 0$, then the real part of both $\lambda_1$ and $\lambda_2$ are negative. Let us therefore assume :

$$(2) - ((3)/(4\sqrt{(1)}\,)) > 0.$$

Then $\lambda_1$ and $\lambda_2$ are real and $\lambda_1 < \lambda_2$, and we only need to show that $\lambda_2 < 0$ :

$$\lambda_2 = -\frac{1}{2}(a_f + a_s) - \frac{1}{2}\sqrt{(1)} + \frac{1}{2}\sqrt{(2) - \frac{(3)}{4\cdot\sqrt{(1)}}}$$

$$= -\frac{1}{2}(a_f + a_s) - \frac{1}{2}((1-\mu)a_s + \mu a_f)\{\chi(\overline{p}_1)\chi(\overline{p}_2)\}^{1/2}$$

$$+ \frac{1}{2}\{ (a_f - a_s)^2 + ((1-\mu)\, a_s + \mu a_f)^2 \chi(\bar{p}_1)\chi(\bar{p}_2)$$

$$- \ 2(a_f - a_s)[\chi(\bar{p}_1)\chi(\bar{p}_2)]^{1/2}((1-\mu)a_s - \mu a_f)) \}^{1/2}$$

$$= -\frac{1}{2}\ (a_f + a_s) - \frac{1}{2}((1-\mu)a_s + \mu a_f)\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2}$$

$$+ \frac{1}{2}\{ \{ a_f(1 + \mu\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2}) + a_s(1 + (1-\mu)\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2})\}^2$$

$$- 2a_f a_s (1 + \{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2}) - 2a_f a_s(1 + \{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2})\}^{1/2}$$

Since the last two terms in this expression is negative, we have

$$\lambda_2 < -\frac{1}{2}(a_f + a_s) - \frac{1}{2}((1-\mu)a_s + \mu a_f)\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2}$$

$$+ \frac{1}{2}\{ \{ a_f(1 + \mu\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2}) + a_s(1 + (1-\mu)\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2})\}^2\}^{1/2}$$

$$= -\frac{1}{2}(a^f + a_s) - \frac{1}{2}((1-\mu)a_s + \mu a^f)\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2}$$

$$+ \frac{1}{2}\{ a_f(1 + \mu\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2}) + a_s(1 + (1-\mu)\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2}) = 0$$

Hence $\lambda_2 < 0$ and both eigenvalues have negative real parts.

Let us now look at the two last eigenvalues $\lambda_3$ and $\lambda_4$ . If (3) is positive then $\lambda_3 < \lambda_4$ and we just have to look at $\lambda_4$. However it is possible that they are complex, if (2) + ((3)/(4 $\sqrt{(1)}$ )) is negative. In this case the real part of $\lambda_3$ is less than the real part of $\lambda_4$ and the real part of $\lambda_4$ is given by:

$$-\frac{1}{2}(a_f + a_s) + \frac{1}{2}\sqrt{(1)} < 0, \text{ when } \chi(\bar{p}_1)\chi(\bar{p}_2) < 1.$$

When (2) + ((3)/(4 $\sqrt{(1)}$ )) is positive then $\lambda_3 < \lambda_4$ and we have to show that $\lambda_4 < 0$:

$$\lambda_4 = -\frac{1}{2}(a_f + a_s) + \frac{1}{2}\sqrt{(1)} + \frac{1}{2}\sqrt{(2) + \frac{(3)}{4 \cdot \sqrt{(1)}}} \Rightarrow$$

(A.1) $\lambda_4 = -\frac{1}{2}(a_f + a_s) + \frac{1}{2}((1-\mu)a_s + \mu a_f)\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2}$

$$+ \frac{1}{2}\{ (a_f - a_s)^2 + ((1-\mu)a_s + \mu a_f)^2 \chi(\bar{p}_1)\chi(\bar{p}_2)$$

$$+ 2(a_f - a_s)\{\chi(\bar{p}_1)\chi(\bar{p}_2)\}^{1/2}((1-\mu)a_s - \mu a_f))\}^{1/2}$$

$\lambda_4$ attains its highest value if $(1-\mu)a_s - \mu a_f > 0$, in this case (3) is positive. Let us therefore assume that $(1-\mu)a_s - \mu a_f > 0$. We assumed that $\chi(\bar{p}_1)\chi(\bar{p}_2) < 1$, then right-hand side of (A.1) is less than

$$< -\frac{1}{2}(a_f + a_s) + \frac{1}{2}((1-\mu)a_s + \mu a_f) + \frac{1}{2}\{ (a_f - a_s)^2 + ((1-\mu)a_s + \mu a_f)^2$$

$$+ 2(a_f - a_s)((1-\mu)a_s - \mu a_f))\}^{1/2}$$

$$= -\frac{1}{2}((1-\mu)a_f + \mu a_s) + \frac{1}{2}\{ ((1-\mu)a^f + \mu a_s)^2 \}^{1/2} = 0.$$

If $\chi(\bar{p}_1)\chi(\bar{p}_2) < 1$, then $\lambda_4 < 0$ and all 4 eigenvalues have negative real parts. ■

## Chapter 7. Conclusions.

In recent literature of dynamic macroeconomic theory, there has been a great interest in the convergence of simple adaptive learning rules to rational expectations solution, but is has been common to assume that agents are homogeneous in their expectation formation, it thus would be natural to allow for some heterogeneity in the agents learning rules.

In this thesis, we have analysed heterogeneous beliefs within the framework of a standard overlapping generations model. This was done to extend some of the local stability results developed in the previous literature, where the economic agents use the same learning rule. Furthermore, we analysed the effects on labour supply, prices and welfare when we change the speed of learning. The issue of interest was whether agents with fast learning rule could do better compared to agents with slow learning rule.

Intuitively, agents with a more accurate forecast should have a higher level of welfare compared to agents with a less accurate forecast. However, this need not be the case as the results in chapter 2, 3 and 4 show. This was due partly to the assumption that both groups of agents were boundedly rational, and there was no interaction between the two groups. In this situation, the fast learners were "closer" to the steady state value of the variable of interest, and depending on the initial conditions, the fast agents could be worse off in the long run, even though they made a "better" forecast.

In the simple overlapping generations model introduced in chapter 2, we were able to shows this learning effect, if both types of agents initially were above the steady state, the slow learners had a higher welfare in short run, but in the long run the fast agents were better off. There was a single crossing point in time, the slow agents had a higher welfare initially, but at some stage the fast became better off than the slow for the rest of the learning transition. The crossing point depends on the choosen parameter values. If the two groups initially were below the steady state, it was possible to show the opposite result. The fast agents were better off in short run, while the slow was better off in the long run. The intuitive reason behind these two results are as follows

- The two economies did not interact, hence the fast (slow) agents was followed by fast (slow) agents. Since there was a delay in revising the expectations initially, then both types of agents supplied the same amount of labour initially. The following generation of fast agents react more rapidly than the slow agent, due to a higher speed of learning, which meant a reduction in the labour supply, when they initially were above the steady state, or an increase when they were below the steady state. Hence the welfare of the old generation of fast learners is lower compared to the old generation of slow agents, because the consumption of the old depends on the labour supply of the young, when the labour supply is reduced, while the opposite applies when the labour supply is increased.
- The level of welfare during the learning transition was below the steady state welfare, but given that the steady state was locally stable under learning, the welfare "increase and converge". Since the fast agents was "closer" to the steady state during the learning transition, because they have a higher speed of learning, they eventually have a higher welfare. If the level of welfare during the learning transition was above the steady state welfare, we have the opposite result.

The two situations, where the economy was either above or below the steady state, could be used in different types of policy experiments. However, government intervention only reduced the steady state level of welfare, and during the learning transition the agents were either "undershooting" or "overshooting" the steady state welfare, depending on the initial conditions.

There was no scope for government intervention in chapter 2. If, there are increasing returns various forms of government intervention can increase the level of welfare in a new steady state. Hence the government can use fiscal and/or monetary policy to steer the economy towards a steady state, or to put it another way, the agents' learning rule takes them to high-level equilibrium after the government intervention has removed the low-level equilibrium. In this case, the fast learners were better of during the entire learning transition as long as the government intervention was not too "large". If the government intervention were "large", the slow could be better off than the fast in the long run, but initially the fast was better off than the slow.

An obvious extension was to incoporate a random shock in this model, for example, a productivity shock. This could generate fluctuations between a low-level equilibrium and a high level equilibrium, even if there was no policy changes or other changes, as long as the agents use a constant speed of learning. In order to obtain local stability under learning, it was necessary to use a decreasing speed of learning, otherwise the forecast is still random even in the limit. The reason for choosing a constant speed was that is was easier to track changes in policy with a constant learning rule. The welfare comparison for two economies did not necessarily give the same results as in chapter 2 and 3.

It would be natural to study the case with constant speeds further, if we want to analyse more complicated changes in government policies, for example, a time-varying policy, and perhaps allow for the agents to forecast on the policy changes as well. We should also try and prove the conjectures made in chapter 3 and 4, and try and find the optimal path of policy, briefly discussed in chapter 3. Since it is possible that there exist sunspots solutions in the model, we could investigate the connection between existence and stability of a sunspot solution and the speed of learning. Evans and Honkapohja (1993a) give an example of an economy that is in a sunspot equilibrium, and show how government intervention can eliminate the sunspot solution so the learning rule takes the agent to the steady state after the intervention. We could investigate how big a the speed of learning combined with government intervention would be needed to eliminate the sunspot. We compared two adaptive learning rules, instead we could compare the adaptive learning rule with another learning rule, in order to show which was fastest.

In the second part, we extended the overlapping generations model, such that it includes both fast and slow learners in the model. This changed the conditions for the steady state to be locally stable under learning, and the stability condition depended in a simple way on the fraction of fast learners. The welfare comparison between fast and slow gave the intuitive result that fast learners were better off than slow agents during the learning transition if the initial condition were the same for both types. This was due to the fast agent being closer to the actual value of the forecasted variable at each point in time, hence they were closer to the true optimisation problem compared to the slow learners. If the initial conditions differed then the slow learners could be better off initially, but for t sufficiently large the fast learners overtake as shown by various simulations.

The results in chapter 5 may give some justification for the remark that is easy to incorporate heterogeneous beliefs into existing models, since both the stability results and welfare comparison gave the results we expected. However, the results may depend crucial on the way we incorporate heterogeneous learning in the model, and this was done in simple way in chapter 5. A natural extension of this would be to endogenise the fraction of learners, and make it time dependent, another possibilty was to let the government be able to change fractions. We could also assume that the fraction was related to the welfare in some way, and that fast and slow are trying increase their proportion at the expense of the other, such that fraction of slow or fast goes towards zero. Another extension would be to introduce a random shock to the model as in chapter 4. We briefly studied the case with n learners, instead of n types, we could have continuum of agents, where some might have rational expectations or perfect foresight

Since the standard overlapping generations model may have equilibria other than steady state, it would be obvious to study the stability of such equilibria under learning, when we have mixed economy of slow and fast agents. An attempt was done in the last chapter, where we analysed the local stability of a 2-cycle under learning in the mixed economy. Although a 2-cycle may not be the most complicated equilibrium, it was not as straightforward as expected to find conditions for the 2-cycle to be locally stable under learning. We found a set of sufficient conditions for local stability of the 2-cycle. This was done by looking at a corresponding differential equation instead of difference equation and followed the outline in Ljung (1977). However, Ljungs results was for a stochastic difference equation, while the difference equation in chapter 6 was deterministic. Thus we proved Ljungs result for this simpler algorithm. The stability condition was equivalent to the stability condition, when there is only one type of agents.

This model could again be extended to include a random variable, but this does not make it any easier to find stability conditions. We could also look at more complicated equilibria, such as a k-cycle, sunspot solution or a stationary dynamic path. However, the dynamics often become quite complicated and we may only be able to use simulations in order to obtain some results. It is easy to see that the models become more complex and formal results are to show, when we introduce some sort of heterogeneity in the agents beliefs, or just vary the speed of learning, and there still remains a lot to analyse in these models.

It is easy to see that the models become complicated and the formal results is hard to show, when we introduce some sort of heterogeneity in the agents beliefs, or just vary the speed of learning, and there still remain a lot to analyse in these models. In this thesis, we were able to show in some cases, agents who make better forecasts do not necessarily enjoy a higher welfare compared to agents who are not so quick to respond to changes. In other cases, though the "quick" agents were better off, as we would expect.

*References.*

Azariadis, C. 1981. "Self-fulling prophesies." *Journal of Economic Theory.* 25, 380-396.

Azariadis, C. 1993. *Intertemporal Macroeconomics.* Basil Blackwell. Oxford.

Bray, M. 1982. "Learning, estimation and the stability of rational expectations equilibria." *Journal of Economic Theory*. 26, 501-526.

Bray, M. and N.E: Savin. 1986. "Rational epectations equilibria, learning and model specification. *Econometrica*. 54, 1129-1160.

Brock, W. A. and A.G. Mallearis. 1989. *Differential Equations, Stability and Chaos in Dynamic Economics.* North Holland. Amsterdam.

Bullard, J. 1994. "Learning equilibria". *Journal of Economic Theory*. 64, 468-485.

Cooper, R. and A, John. 1988. "Coordinating coordination failures in keynesian models." *Quarterly Journal of Economics*. 113, 441-464.

Duffy, J. 1994. On learning and the nonuniqueness of equilibrium in an overlapping generations model with fiat money. *Journal of Economic Theory*. 64, 541-553.

Evans, G.W. 1983. The stability of rational expectations in macroeconomic models. in R. Frydman and E.S. Phelps, eds. *Individual forecasting and aggregate outcomes : Rational expectations' examined.* Cambridge University Press. Cambridge.

Evans, G.W. 1989. " The fragility of sunspots and bubbles". *Journal of Monetary Economics*. 23, 297-317.

Evans, G. W. and S. Honkapohja. 1993a. "Learning and economic fluctuations." *European Economic Review*. 37, 595-602.

Evans, G. W. and S. Honkapohja. 1993b. "Adaptive forecasts, hysteresis and endogenous Fluctuations." *Federal Reserve Bank of San Francisco Economic Review*. 1993 (1), 1071-1098.

Evans, G. W. and S. Honkapohja. 1994a. "Learning, convergence, and stability with multiple rational expectations equilibria." *European Economic Review*. 38, 1071-1098.

Evans, G. W. and S. Honkapohja. 1994b. "On the local stability of sunspot equilibria under adaptive learning rules". *Journal of Economic Theory*. 64, 142-161.

Evans, G. W. and S. Honkapohja. 1995a. "Local convergence of recursive learning to steady states and cycles in stochastic nonlinear models". *Econometrica*. 63, 195- 206.

Evans, G. W. and S. Honkapohja. 1995b. "Increasing social Returns, learning, and bifurcation phenomena." in A. Kirman and M. Salmon, eds. *Learning and Rationality in Economics*. Basil Blackwell, Oxford, 216-235.

Evans, G. W. and S. Honkapohja. 1995c. "Adaptive learning and expectational stability: An introduction." in A. Kirman and M. Salmon, eds. *Learning and Rationality in Economics*. Basil Blackwell, Oxford, 102-126.

Evans, G. W. and S. Honkapohja. 1995d. "Economic dynamics with learning: New stability results. Mimeo.

Evans, G.W., S. Honkapohja and T.J. Sargent. 1993. "On the preservation of determinstic cycles when some agents perceive them to be random fluctuations." *Journal of Economic Dynamics and Control* 17. 705-721.

Evans, G. W, S. Honkapohja and R. Marimon. 1995. "Convergence in monetary models with heterogeneous learning rules." Mimeo.

Frydman, R. 1982. Towards an understanding of market processes. *American Economic Review* 72. 652-668

Goldberg, S. 1958. *Introduction to Difference Equations*. John Wiley and Sons. New York.

Grandmont, J.M. 1985. "On endogenous competitive business cycles." *Econometrica*, 53 . 995-1045.

Grandmont, J. M. and G. Laroque. 1986. " Stability of cycles and expectations." *Journal of Economic Theory*. 40. 138-151.

Grandmont, J. M. and G. Laroque. 1991. " Economic dynamics with learning : Some instability results." in *Equilibrium Theory and Applications*, Proceedings of the Sixth International Symposium in Economic Theory and Econometrics, ed. by W.A. Barnett et al. Cambridge University Press. 247-273.

Guesnerie, R. and M. Woodford. 1991. " Stability of cycles with adaptive learning rules." in *Equilibrium Theory and Applications*, Proceedings of the Sixth International Symposium in Economic Theory and Econometrics, ed. by W.A. Barnett et al. Cambridge University Press. 111-134.

Guesnerie, R. and M. Woodford. 1992. " Endogenous Fluctuations." *Advances in Economic Theory*: Sixth World Congress. Vol. 2. Editor J.J. Laffont. Cambridge University Press.

Howitt, P and R.P, McAfee. 1992. "Animal spirits." *American Economic Review*. 82, 493-507.

Kiyotaki, N.1988. "Multiple expectational equilibria under monopolistic competition." *Quarterly Journal of Economics*. 103, 695-713.

Kreps, D. 1990. Game Theory and Economic Modelling. Oxford University Press.

Ljung, L. (1977). "Analysis of recursive stochastic algorithms." *IEEE Transactions on Automatic Control*, AC-22, 551-575.

Lucas, R.E. Jr. 1986. " Adaptive behaviour and economic theory." *Journal of Business*. 59. S401-426.

Marcet, A. and T.J. Sargent. 1989a. " Convergence of least squares learning mechanisms in self-referential stochastic models." *Journal of Economic Theory*. 48, 337-368.

Marcet, A. and T.J. Sargent. 1989b. " Convergence of least squares learning in enviroments with hidden state variables and private information." *Journal of Political Economy*. 97, 1306-13022.

Ng. Y-K. 1980. "Macroeconomics with non-perfect competition." *Economic Journal*. 90, 598-610.

Nikaido, H 1968. *Convex structures and Economic Theory*. Academic Press. New York.

Pagano, M. 1990. Imperfect competition, underemployment equilibria, and fiscal policy. *Economic Journal*. 100, 440-463.

Romer, P.M. 1986. " Increasing returns and long-run growth." *Journal of Political Economy*. 94, pp. 1002-37.

Rudin, W. 1964. *Principles of mathematical analysis*. Second edition. McGraw-Hill. New York.

Sargent, T.J. 1993. *Bounded Rationality in Macroeconomics*. Oxford University Press. Oxford.

Sydsaeter, Knut. 1981. *Topics in Mathematical Economics*. Academic Press, London.

Takayama, A. 1985. *Mathematical Economics.* Cambridge University Press.

Townsend, R. 1983. Forecasting the forecasts of others. *Journal of Political Economy.*

Vives, X. 1993. " How fast do rational agents learn ? " *Review of Economic Studies*. 60, pp. 329-347.

Woodford, M. 1990. " Learning to believe in sunspots. *Econometrica*. 58, 277-307.

**The computer programs used for the simulations in chapter 2-5.**

The computer programs are made in *Mathematica for Windows 2.2*. A program for the simulations in chapter 2 is made as follows.

The temporary equilibrium condition :

**F[n_] : = B*n^(beta).**

The learning rule :

**h[n_] : = n + a*(F[n] – n).**

In order to generate a sequence of expected labour supply, we choose parameter values for **B**, **beta**, **n** and an initial condition. The following command give a sequence of expected labour supply:

**NestList[ h, 1, 100 ]**

This command give the following output

**h [1] , h [ h[1] ] , h [ h[ h[1] ] ] , .........., h [ [ .....h[1]......] ]**

Here the initial condition is 1, and we have sequence with 100 values for the expected labur supply. This can now be used to calculate the welfare sequence since $U(c_{t+1}) - V(n_t)$ could be written as a function of the expected labour supply.

**u[n] – v[n] = u[ f [ F[h[n]] ] ] – v[ F[n] ]**

The sequence of expected labour supplies can then be mapped in the this expression and we have a welfare sequence. The good thing about *Mathematica* is that we can use symbols and then put in values for the symbols, so it is easy to change the parameters. *Mathematica* has a command that can plot the welfare sequence. This gives the figures in chapter 2. The rest of simulations is extensions of these simple commands.

In chapter 4 we use a decreasing speed of learning this change the **h**-function to

**h[n_,t_] : = n + (a/t)*(F[n] – n).**

**p[n_,t_] : = t+1.**

In this case we need to update to functions **h** and **p**. We define

```
Iteration[x_] := { h[ x[[1]],x[[2]] ], p[ x[[1]],x[[2]]] }
NestList[ Iteration , {5,1}, 100 ]
```

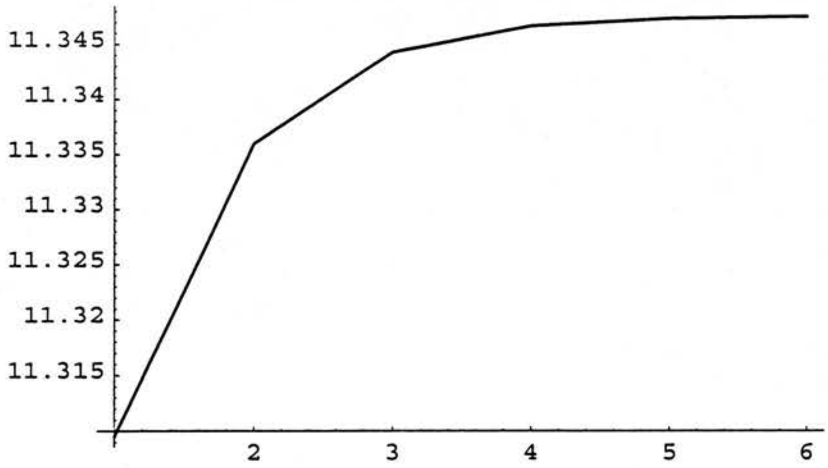
This gives us a sequence of expected labour supplies. In this way we define function of n variables and make sequences. The same method is used in chapter 5.

```
A
A
alpha
alpha
gamma
General::spell1:
   Possible spelling error: new symbol name "gamma"
     is similar to existing symbol "Gamma".
gamma
sigma
sigma
epsilon
epsilon
a
a
B
B
beta
General::spell1:
   Possible spelling error: new symbol name "beta"
     is similar to existing symbol "Beta".
beta
F[n_]:= B*(n)^beta
F[n]
   beta
B n
B = (alpha*(A^(1-sigma)))^(1/(1+epsilon))
  1 - sigma        1/(1 + epsilon)
(A          alpha)
beta = alpha*((1-sigma)/(1+epsilon))
alpha (1 - sigma)
-----------------
   1 + epsilon
F[n]
  1 - sigma        1/(1 + epsilon)
(A          alpha)

  (alpha (1 - sigma))/(1 + epsilon)
 n
h[n_]:= n + a*(F[n] - n)
u[n_]:= (1/(1-sigma))*(A*((F[h[n]])^alpha))^(1-sigma)
v[n_]:= (1/(1+epsilon))*(F[n])^(1+epsilon)
welfare[n_]:= u[n]-v[n]
gamma = 0
0
```

```
A= 28
28
a=0.8
0.8
sigma = 0.41
0.41
epsilon = 0.2
0.2
alpha = 0.2
0.2
F[n]
        0.0983333
1.346 n
h[n]
               0.0983333
0.8 (1.346 n            - n) + n
welfare[n]
          0.118
-1.19034 n      + 12.5371

                  0.0983333           0.0116033
  (0.8 (1.346 n            - n) + n)
NestList[h, 2, 5]
{2, 1.55275, 1.43496, 1.40272, 1.39378, 1.39129}
Map[welfare, %]
{11.3095, 11.336, 11.3443, 11.3467, 11.3474, 11.3476}
ListPlot[%, PlotJoined -> True]
```




-Graphics-